ELEMENTARY SURVEYING

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FOURTH EDITION

D. VAN NOSTRAND COMPANY, INC.

PRINCETON, NEW JERSEY

TORONTO NEW YORK LONDON

D. VAN NOSTRAND COMPANY, INC.
120 Alexander St., Princeton, New Jersey (*Principal office*)
24 West 40 Street, New York 18, New York

D. VAN NOSTRAND COMPANY, LTD.
385, Kensington High Street, London, W.14, England

D. VAN NOSTRAND COMPANY (Canada), LTD.
25 Hollinger Road, Toronto 16, Canada

Published simultaneously in Canada by
D. VAN NOSTRAND COMPANY (Canada), LTD.

First Edition January 1937

Four Reprintings

Second Edition, July 1943

Thirteen Reprintings

Third Edition, March 1955

Three Reprintings

Fourth Edition, January 1963

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

This textbook is designed to meet the needs of (1) the beginning course in surveying and mapping for civil engineering students; (2) an elementary course in surveying for students in junior colleges and city colleges where some basic science and engineering are taught; (3) a course to provide the necessary surveying and mapping knowledge for students in geology, geography, forestry, landscape architecture, and agriculture.

This fourth edition of Elementary Surveying is characterized by three important features. (1) A chapter on photogrammetry, sufficiently developed with theory, tables, and solutions, demonstrates the determination of the three-space coordinates of the ground points appearing in the overlapping areas of aerial photographs. (2) The treatment of all subject matter is brought up to date by the description of the new instruments and methods that are now available. (3) The book appears in a new format that provides a larger page, more legible print, and better illustrations.

Examples of subjects in which there have been recent developments in the theory and practice of surveying and which receive attention include (1) the new instruments, including the optical transit, automatic level, subtense bar, tellurometer, geodimeter, and electrotape; (2) the partial displacement of triangulation by trilateration; (3) alternate methods of reducing Polaris observations; (4) the use of state-wide plane coordinates in land surveying; (5) the new isogonic chart, and (6) construction surveying.

Answers are given to many of the assigned problems.

Like its predecessors, this revision maintains brevity of treatment consistent with completeness and clarity, so essential to effective teaching. Also the subjects of errors and checks that are fundamental to all surveying measurements are given careful consideration throughout.

Grateful acknowledgement and thanks are extended to colleagues and instrument makers who have given valuable suggestions and assistance.

W.H.R.
M.O.S.

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CHAPTER 1

GENERAL PRINCIPLES

1-1. Surveying Defined Surveying is the art of determining the positions of points on or near the earth's surface by means of measurements in the three elements of space; namely, distance, direction, and elevation. The dimensions of distance and elevation are measured in units of length, while that of direction is measured in units of arc. Hence, it may be said that all surveying operations comprise the measurement of distances (horizontal and vertical) and of angles.

The purposes of most surveys require the computation of areas and volumes, and the representation of the field measurements on profiles and maps. Therefore, these procedures are properly considered a part of the surveyor's work.

1-2. Kinds of Surveys The shape of the earth is that of an oblate spheroid of revolution, the polar axis being somewhat shorter than the equatorial. When surveys are of such wide extent that the spheroidal shape of the earth is a matter of importance, they are called *geodetic surveys.* When they are of such limited extent that the exact shape of the earth may be disregarded, they are called *plane surveys.* The surveys with which most engineers are concerned are of the latter kind.

Topographic Surveys are those which are made for the purpose of representing the three-dimensional relations of the earth's surface on maps or models. The features shown include such natural objects as streams, lakes, timber, relief of the ground surface, etc.; and the works of man, such as buildings, roads, railways, cultivation, towns, and villages. The devices used to represent the relief of the earth's surface include hachures, shading, contours, and models.

Land Surveys are those which are made incident to the fixing of property lines, the calculation of land area, or the transfer of real property from one owner to another. Land surveys have their origin in the beginnings of civilization.

Route Surveys are made for the purpose of locating and constructing engineering projects which are built along fixed routes and gradients. These include highways, railways, canals, drainage ditches, levees, and transmission lines.

Hydrographic Surveys comprise the operations necessary to map the shore lines of bodies of water; to chart the bottom areas of streams, lakes, harbors, and coastal waters; to measure the flow of streams, and to determine other factors affecting navigation and the water resources of the nation.

Mine Surveys are necessary to determine the position of all underground workings and surface structures of mines, to fix the positions and directions of tunnels, shafts, and drifts, and to fix the surface boundaries of all claims, and properties.

Cadastral Surveys are made to fix the boundaries of municipalities and of state and federal jurisdictions.

Aerial Surveys are those which make use of photographs taken from airplanes. Such photographs may be used in connection with any of the classes of surveys metioned above and serve a variety of purposes. The results are used in the form of oblique views of the landscape, mosaics of matched vertical photographs, and topographic maps drawn from the detail shown in the photographs. The aerial survey method has important advantages and is being widely and extensively used.

Industrial Surveying, or *optical tooling,* as it is more frequently termed, refers to the use of surveying techniques in the aircraft and other industries where very accurate dimensional layouts are necessary. Modified versions of conventional surveying equipment, such as the *jig transit,* have been designed especially for optical tooling.

1-3. Classes of Surveys All of the above kinds of surveys may be classified in three groups; namely, *land surveys, construction surveys,* and *informational surveys.*

Land Surveys have been characterized in the preceding article.

Construction Surveys.—Inasmuch as all engineering works are built on or below the earth's surface, measurements are necessary properly to plan and construct them. This class comprises the larger

part of all surveys with which the engineer and architect are concerned.

Examples of such surveys are the route surveys for transmission lines, highways, railways, etc.; the site surveys for dams, reservoirs, buildings, bridges, etc.; and surveys for public improvements such as water works, street pavements, and many others.

Informational Surveys.—Many surveys are made for the purpose of constructing maps or charts only, and of fixing accurately the positions of widely separated points (called control points) on the earth's surface. Such surveys serve a wide variety of needs; important among these are the execution of governmental functions and the study, by all concerned, of the natural resources of the nation. Since these surveys are of wide extent and for the general benefit of the public, they are executed by governmental organizations.

The most important Federal surveying and mapping agencies are as follows:

(*a*) *U.S. Geological Survey.*—This organization, which was established in 1879, is charged with the responsibility for preparing the National Topographic Map Series covering the United States and its outlying areas.

(*b*) *U.S. Coast and Geodetic Survey.*—This Bureau celebrated the completion of 150 years of service in 1957. It publishes nautical charts of the coastal waters of the United States and its territorial possessions, executes the principal geodetic surveys of the country, and prepares and distributes the aeronautical charts needed by American civil aviation.

(*c*) *U.S. Navy Hydrographic Office.*—This agency performs essentially the same hydrographic charting functions as the Coast and Geodetic Survey but with respect to waters not contiguous to the United States and its possessions.

(*d*) *U.S. Lake Survey.*—This is the nautical charting agency of the Corps of Engineers. It is concerned primarily with the publication of navigation charts for the Great Lakes.

(*e*) *U.S. Bureau of Land Management.*—This organization is responsible for surveys of the public domain. Rectangular public surveys are still being executed in some of the Western States and in Alaska.

1-4. Historical Notes *Ancient Times.*—The early history of surveying merges with that of mathematics, and in fact the theory

of mathematics seems to have grown out of the practical use of numbers and calculations required in the life of the community, examples of which were the establishment of boundaries and the measurement of land areas. This relation between mathematics and surveying is indicated by the term applied to one of the earliest branches of mathematics, namely, *geometry*, which is derived from Greek words meaning "earth measurements."

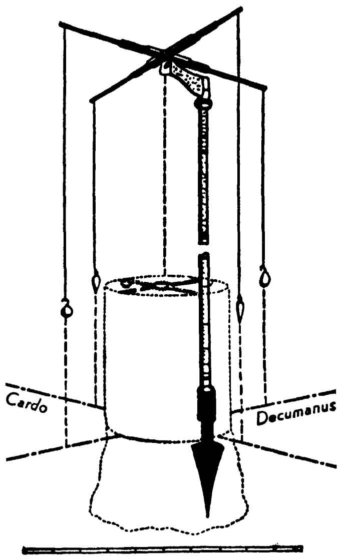


FIG. 1-1. Roman Groma.

Roman Instruments.—The Roman surveyors were known as *gromatici* from their use of the *groma*, Fig. 1-1. It consisted merely of crossarms fixed at right angles and pivoted eccentrically upon a vertical staff. From the ends of the arms plumb lines were suspended. The whole purpose and use of the instrument was to establish upon the ground two lines at right angles to each other.

The *chorobates* was an instrument for leveling. It was a wooden bar about 20 ft long, in which a groove was made at its middle about one inch deep and 5 ft long. When the bar was leveled so that water stood evenly in the groove, a horizontal line was established.

The Telescope.—The discovery of the telescope is generally accredited to Lippershey, about 1607. This invention when applied to surveying instruments is of the greatest importance in increasing the precision and speed of surveying measurements.

The Vernier.—In 1631 Pierre Vernier, a Frenchman, published in Brussels the description of his device which, bearing the inventor's name, is in general use today as a means of accurately determining subdivisions of a graduated scale, Figs. 3-13, 5-2.

The Semicircumferentor.—Before the telescope was applied to angle measuring instruments, the peep sight served to fix the line of sight on most instruments used in mine and land surveying. An example of this type is shown in Fig. 1-2.

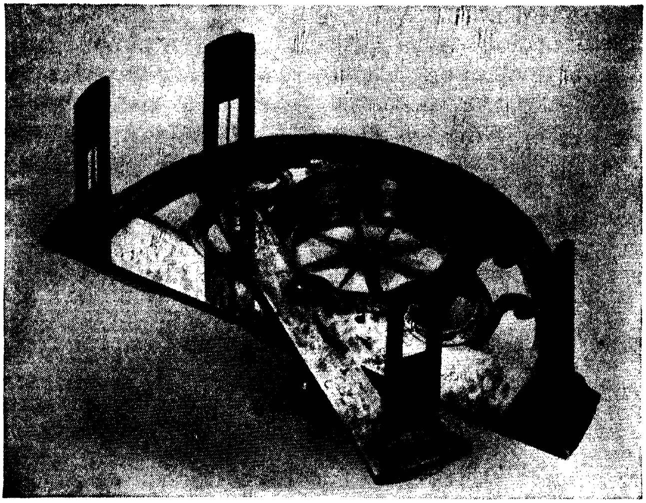


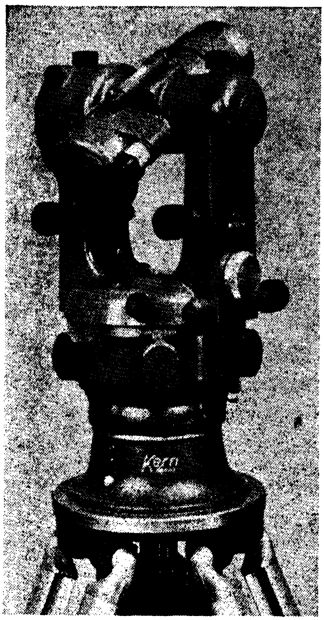
FIG. 1-2. Semicircumferentor.

Courtesy of Keuffel & Esser Co.

FIG. 1-3. First Transit Made in the United States. It was manufactured in Philadelphia in 1831 by William Young.

Instruments of this type in the hands of George Washington, Thomas Jefferson, Mason and Dixon, and others of less renown, served to lay down the boundaries of the colonial possessions and boundaries of the thirteen original states, and to locate our canals and railroads in the early period of their development.

The Transit.—Two men, Draper and Young, working independently, in the early years of the nineteenth century (about 1830) brought together in one instrument, Fig. 1-3, the essential parts of what has long been known as a "transit," and which with many later improvements has been recognized as the surveyor's most important instrument, Fig. 1-4. A modern transit of high precision is more properly called a "theodolite."



Courtesy of Kern Instruments, Inc.

FIG. 1-4. Modern Engineer's Transit.

Units of Length.—The commentaries of Caesar have made us familiar with the *mille passum* of the Roman armies, from which our terms "mile" and "pace" are clearl derived; also the *ager* or area of land that could be plowed in one day by a yoke of oxen, has become our "acre." Again the Roman *pertica,* meaning a measuring stick or pole, became in English usage the "perch," "pole," "rood" or "rod" still used in our term "level rod," also as a unit of length ($16\frac{1}{2}$ ft) in land measure.

The necessity of standardizing the length of the rood was soon recognized, and in his book on surveying (1570) Koebel gives the following method: "A rood should by the right and lawful way, and in accordance with scientific usage, be made thus: sixteen men, short and tall, one after the other, as they come out of church (Fig. 1-5) should place each a shoe in one line; and if you take a length of exactly 16 of these shoes, that length shall be a true rood."

The area of an acre of land was established in England by an act of the Star Chamber in the 12th year of the reign of Henry VII,

which "setteth down that an acre should be XL pole in length and IV pole in breadth." This was followed by an Act of Queen Elizabeth: "Commons or waste grounds lying within 3 miles of London shall not be enclosed. A mile shall contain 8 furlongs, every furlong 40 poles, and every pole shall contain 16 foot and an half."

FIG. 1-5. Determining the Length of a Rood.

To render land measure into a decimal system, Gunter devised a chain 66 ft long having 100 links. The link chain is now obsolete, but steel tapes, etched with these units, are still in use.

1-5. Surveying Theory Plane surveying makes extensive use of geometry, algebra, and trigonometry. A thorough knowledge of these subjects is expected of the student when he begins, or at least before he completes, his study of elementary surveying practice. Advanced surveying makes use of higher mathematics, so that the special work of geodetic surveying requires the services of qualified mathematicians.

However, although the theory of plane surveying is quite simple, its practice requires those qualities of resourcefulness, good judg-

ment, and general ability in such degree that the best efforts are demanded of the student who would become proficient. Because of these conditions, the student will find that the study of surveying provides not only practical and useful information, but an excellent opportunity to develop such qualities as self-reliance, initiative, ability to get along with others, etc., which are most essential to successful engineering practice.

1-6. Surveying Measurements All surveying operations are subject to the imperfections of the instruments and of the errors inherent in their manipulation. Therefore, *no surveying measurement is exact.* Accordingly, the nature and magnitude of the errors in the surveyor's work must be well understood if good results are to be secured.

Obviously, there are many degrees of precision possible in any measurement. Thus, the distance between two established fence corners may be measured by estimation, by pacing, by stadia, or by taping. Each of these methods may be the best one to use for a given purpose because, ordinarily, it is a waste of time and money to attain unnecessarily high accuracy. On the other hand, a faulty survey results if the measurements are not sufficiently precise.

The best surveyor then, is not the one who makes the most precise measurements, but he who is able to choose and apply just that degree of precision requisite to his purpose.

1-7. Checks Because of the inherent errors and mistakes in surveying measurements, it is necessary to apply checks to the results (1) to detect mistakes, and (2) to determine the degree of precision attained. Since there is no assurance that any original measurement is free from mistakes, and since any important mistake is intolerable in final results, it is necessary that all work shall be checked; and, no work, either field or office, may be considered complete until it has been checked. It is difficult to convey the importance of this matter to the novice. Invariably, the engineer in charge of surveys of any magnitude is more concerned about the mischief caused by undetected mistakes than he is by the inherent errors of the measurements.

1-8. Field Notes No phase of the surveyor's work is of greater importance, or requires more careful attention, than keeping the

field notes. The student should realize from the beginning that the quality of his work is reflected directly from his field record.

Characteristics.—The field notes constitute the only reliable record that exists of the measurements and other items of pertinent information that are obtained in the field. Consequently, they must be recorded in the field, and they should be permanent, legible, complete, and capable of but one, and that the correct, interpretation.

The notes should be kept in a bound or loose-leaf book of good quality paper and recorded with a fairly hard (4H) drawing pencil. The pencil point should be sharp and sufficient pressure used to indent the paper slightly.

Field notes are recorded in the field. Records made on scratch paper and copied later, or other data recorded from memory, may be useful, but they are not field notes. The authority and reliability of field records are always under suspicion unless they have been entered in the book at the time and place the data were obtained.

To be complete, the notes should show all data, together with a sufficient interpretation to answer all questions that may properly be raised with respect to any given survey. Such completeness cannot be achieved unless the surveyor keeps clearly in mind not only the immediate uses of the data, but those which may resonably be expected to arise at some future time. It often becomes desirable or necessary to retrace or extend surveys after many years, and after many physical changes have been made in the vicinity. The original field notes afford the only means of accomplishing the desired ends, and unless they are sufficiently complete, they may be worthless.

If field notes are to be useful, they must be legible. To this end they should be lettered instead of written. The Reinhardt style is generally used. This style and suggestions regarding this matter are given in Chapter 14. Field notes are subject to the hard usage of field conditions, and the pages may become soiled or wet. For this reason the hard pencil and sharp point are necessary.

The lettering should be of such size as to permit a reasonable amount of data to be entered on the page without crowding. A suitable size is that in which capitals, numerals, and stems of lower-case letters are made equal in height to one half the space between lines. It should be remembered, however, that compared with the other costs of a survey, the notebook expense is very slight, and plenty of space should be used.

To avoid any possible misinterpretation of the notes, the note-

keeper must know the purpose for which the survey is undertaken, and must keep the point of view of the one who is to use the notes, and who, probably, was not present in the field when the survey was made.

Standard forms are provided for tabulating the observations for many of the common surveying methods, but in many cases the surveyor must devise his own arrangement of the recorded data. In all cases, explanatory remarks will be necessary to make clear the field procedure in all its details. To this end, field sketches are very useful and should be used freely. They are not drawn to scale usually, but the relations in such sketches should be true within the limits of the conditions under which they are made.

Two Important Principles.—Two general principles apply to all field notes. (1) Great care must be exercised to explain in detail how each survey begins and how it ends. The data which pertain to the intermediate steps are more easily interpreted either in the form of notes or by the knowledge of the methods in use, but the record of the manner of beginning and ending the survey is usually not clear to anyone other than the note-keeper, and it is most essential to any proper use of the notes. This will usually require a paragraph of explanation, sketches, and perhaps cross-references to data on other pages in the book.

(2) The final results and all checks obtained should be plainly indicated.

Erasures.—A bona fide record of a field measurement should never be erased. Such erasures always cast doubt on the reliability of the changed records. Further, what may at one time appear to be an incorrect value, may at a later time be found to be correct and if it has been erased, a repeated measurement may be necessary.

If it is determined that the original record is incorrect, a line is drawn through it and the correction is made above it. Of course, such obvious and nonpertinent mistakes as a misspelled word, the wrong heading for a column, or change of wording in a sentence, may properly be erased.

1-9. Errors It has been stated previously that no surveying measurement is exact. The surveyor, therefore, is necessarily and continuously dealing with errors. If his work is to be well done, he must understand thoroughly the nature of the sources and behavior of the errors which affect his results.

This chapter presents briefly some of the general principles of this important subject, and, although the treatment will be quite elementary throughout, the discussion will be continued in connection with nearly every subject in this book. It is hoped that this continued repetition will not only emphasize the importance of the subject, but will materially aid the beginner in the development of good judgment in his work.

1-10. Classes of Errors The sources of error in surveying measurements may be classified under three groups—instrumental, personal, and natural.

Instrumental Errors are those due to imperfections in the instruments used, either from faults in their manufacture, or from improper adjustments between the different parts. The incorrect length of a steel tape, or the improper adjustment of the plate bubbles of a transit are examples. It is understood that the instruments are never perfect; therefore, proper corrections and field methods are applied to bring the measurements within the desired limits of precision. It may be added that some of the principal advances in the art of surveying within recent years have been effected by improvements in the design and manufacture of the instruments.

Personal Errors arise from the fallibility of the senses of sight and touch on the part of the observer. He must frequently estimate fractional parts of scale divisions, and he must manipulate the instruments with dexterity. These operations are never done exactly, and the magnitude of the resultant errors depends largely on the coordination of the senses of sight and touch, and on the skill of the observer. Reading the divisions on a graduated circle or fixing the line of sight of a transit on a given object are examples.

Natural Errors have their sources in the phenomena of nature, such as changes in temperature, differential refraction of the atmosphere, wind, curvature of the earth, etc. Such sources of error are quite beyond the control of the observer, but he can take proper precautions and adapt his methods to conditions so as to keep the resulting errors within proper limits.

1-11. Definitions of Errors In surveying measurements two kinds of errors are effective, namely, *systematic* and *accidental* errors. These are to be defined in terms of their behavior as to sign

and magnitude both when the field conditions are constant and when the field conditions are changing.

Systematic Error.—A systematic error is one which (a) for constant field conditions is constant both in sign and magnitude; and which (b) for changing field conditions is constant in sign but variable in magnitude, proportionate with the changing field conditions. Hence, the total error resulting from a series of systematic errors is the sum of the separate errors.

An example is the error resulting from the thermal expansion of a steel tape. It is assumed that the tape has a known length at a given (standard) temperature. If a distance is measured at a different, but constant, temperature the resulting error will be constant in magnitude and cumulative.

Accidental Error.—An accidental error is one (a) whose sign is just as likely to be plus as minus under both constant and changing field conditions; (b) the magnitude of the separate accidental errors under all conditions vary according to the laws of chance; and (c) because of the changing signs and magnitudes of the errors their total effect on a measurement will be compensative, as stated in the *law of compensation* in Art. 1-15.

An example is that of setting pins at the end of the tape when measuring distance. The attempt is made to set the pin exactly opposite the end mark on the tape, but for a given measurement there is an equal probability that the pin will be set either beyond, or short of, the end mark, thus causing either a plus or a minus error, and the magnitude of the error has no proportional relation to the field conditions. Accordingly, the signs of the separate errors are either plus or minus, the magnitudes vary according to the laws of chance, and their total effect is compensative.

Discrepancy.—A discrepancy is the difference between two measurements of a given quantity. Usually, although not always, it indicates the precision with which the measurements have been made.

Mistakes.—Mistakes are gross differences from true values due to carelessness or inattention on the part of the observer. They are detected by checking results.

An example is that of misreading the numbers on a tape. Thus, a measured distance of 76 ft may be misread and recorded as 79 ft Such a mistake may be detected by a duplicate measurement in which, presumably, the mistake will not be repeated; or other known

conditions in the field may serve as a check and reveal the presence of the mistake.

1-12. Distinction Between Errors and Mistakes The distinction between errors and mistakes is important, because in dealing with the various sources of error, the subject of mistakes is disregarded. It is always presumed that proper checks have been applied and all important mistakes removed from final values.

The distinction arises from the fact that errors result from sources that cannot be avoided, whereas mistakes can be avoided by paying careful attention to the work in hand. The effects of errors can be minimized, but they cannot be entirely eliminated; whereas mistakes can be detected and removed from final results.

Although mistakes can be avoided, we know from experience that such perfection is not to be expected. "To err is human," and it is not necessarily discreditable on the part of an engineer occasionally to make a mistake. But, it may be discreditable to permit anyone else to discover it, because, if his work is thoroughly done, he will be careful to apply all necessary checks to insure that his results are free from mistakes. Of course, if they are repeated too frequently they indicate an attitude that can be condoned only within limits. Accordingly, because of his sense of pride in his work and a due consideration of his professional reputation, the competent engineer will not be content with his work until all serious mistakes have been detected and removed.

1-13. Distinction Between Precision and Accuracy It is necessary to distinguish between the meanings of the terms *precision* and *accuracy* as they are used in describing physical measurements and the subsequent computations.

Precision refers to the care and refinement with which any physical measurement is made. It relates to the expertness of manipulation on the part of the observer or to the capabilities of the instrument used.

Accuracy refers to the difference between the final measured value of a quantity and its absolute, or true, value.

The distinction can be illustrated by an example. Consider two 12-in., boxwood, engineers' scales of similar construction, one of which (scale A) has been kept in a damp place so that it has ex-

panded 0.1 in. over its full length, and the other one (scale B) has very nearly its true length, say within 0.001 in. It is evident that if the same care and refinement are used in scaling a given dimension with each of these scales, the result obtained with scale B will be nearer the true value; in other words, of the two dimensions scaled with the same *precision*, one is more *accurate* than the other. Also, it may easily be imagined that a measurement with scale A may be more *precise*, but less *accurate*, than the measurement with scale B.

1-14. Probable Error In order to determine the degree of confidence the engineer may place in the results of certain physical measurements, such as of length, he may find it convenient to calculate a numerical index which is expressive of the precision that has been obtained. Again, it is important not to confuse precision with accuracy. Precision is associated with the accidental errors surrounding the measurement, and its magnitude can be closely approximated provided that all systematic errors have been substantially eliminated and a reasonable number of repeated measurements have been made.

If the arithmetic mean of a number of measurements of the same quantity be considered to be its most probable value, and the difference between each individual measurement and the mean value is calculated, there will result a series of deviations, or *residuals*. If these values, which are also called errors, are arranged in order of magnitude with each error being written as many times as it occurs, it will be possible to ascertain the *middle error*. This is the error that occupies the middle place in the array of errors. That is, one half of the errors are greater than it and the other half are less than it. The designation *probable error* has been given to this middle error. It is to be emphasized that the probable error is not the most likely error. The probable error is that error for which the probability of making an error greater than it is just equal to the probability of making an error less than it. It will be helpful to remember a large probable error indicates a wide dispersion of measured values whereas a small probable error will be expressive of comparatively little scatter of the observed quantities.

The value of the probable error is calculated with the use of the following expressions which are derived in standard textbooks on probability and least squares:

$$E_s = 0.6745\sqrt{\frac{\Sigma v^2}{(n-1)}} \tag{1-1}$$

$$E_m = \frac{Es}{\sqrt{n}} \tag{1-2}$$

in which E_s is the probable error of any single measurement of a series, v is the difference (residual) between any single observation and the mean of the series, n is the number of observations, and E_m is the probable error of the mean.

A typical calculation of the probable error of taping a baseline is shown. The probable error is useful in determining the comparative reliability of various sets of measurements and plays an important role in the accuracy specifications of certain high quality surveys.

EXAMPLE 1-1.

Probable Error of Baseline

	Length (ft)	v	v^2
	1571.30	0.01	0.0001
	1571.27	0.04	0.0016
	1571.30	0.01	0.0001
	1571.33	0.02	0.0004
	1571.35	0.04	0.0016
Mean =	1571.31		Σ = 0.0038

$$E_s = 0.6745\sqrt{\frac{0.0038}{4}} = \pm 0.021 \text{ ft}$$

$$E_m = \frac{0.021}{\sqrt{5}} = \pm 0.009 \text{ ft}$$

1-15. The Law of Compensation As regards accidental errors, it has been found by experience that their behavior conforms to a principle which may be called the *law of compensation.* This may be stated as follows: *in a series of observations, the uncompensated or residual error is proportional to the square root of the number of opportunities for its occurrence.*

An example will illustrate further the distinction between systematic and accidental errors. Consider again the two sources of error mentioned previously—thermal expansion of a steel tape and the setting of pins in taping. Suppose that the distance between two

section corners one mile apart has been measured and that the magnitude of each separate tape error was the same; i.e., the temperature was such that the 100-ft tape was 0.01 ft longer than its standard length, and that the average error in setting pins was also 0.01 ft. In a distance of one mile there will be 53 opportunities for each of these errors to occur, and the resultant errors from these two sources are calculated as follows:

The error due to temperature $= 0.01 \times 53 = 0.53$ ft.
The error due to setting pins $= 0.01 \times \sqrt{53} = \pm 0.07$ ft (nearly).

Thus it appears that for these two sources of error, each of which was of equal magnitude in each separate measurement, after 53 opportunities for the errors to occur, the magnitude of the systematic error was seven times as great as the total estimated value of the accidental error.

1-16. Interrelationship of Errors When a computation involves quantities that are subject to accidental errors of measurement, it is desirable to know how the final result is affected by these errors. The theory of errors permits statements of these relations and, without including the derivations here, two general principles are given.

Summation of Errors.—In a summation of quantities, each of which is affected by accidental errors, the accidental error of the sum is given by the square root of the sum of the squares of the separate accidental errors arising from the several sources. Thus,

$$E_T = \sqrt{E_1^2 + E_2^2 + E_3^2 + \ldots + E_n^2} \qquad (1\text{-}3)$$

in which E_T is the total accidental error and E_1, E_2, etc. are the separate accidental errors affecting the measurement.

EXAMPLE 1-2 In leveling across a river it is estimated that the error in reading the rod is ± 0.05 ft and that the error due to the bubble being off center is ± 0.03 ft. It is desired to calculate the total accidental error, due to these sources alone, of the reading on the distant rod.

$$E_T = \sqrt{(0.05)^2 + (0.03)^2} = \pm 0.06 \text{ ft}$$

Product of Errors.—If a measured quantity, P, is the product of

two independently measured quantities A and B, the accidental error of the product E_P is given by

$$E_P = \sqrt{A^2E_B^2 + B^2E_A^2} \tag{1-4}$$

where E_A and E_B represent the accidental errors of A and B respectively.

EXAMPLE 1-3 The dimensions of a rectangular lot and their estimated uncertainties are as follows:

$$A = 122.30 \pm 0.04 \text{ ft} \qquad B = 48.30 \pm 0.03 \text{ ft}$$

It is desired to determine the estimated error in the calculated area. From Eq. (1-4),

$$E_p = \sqrt{122.30^2 \times 0.03^2 + 48.30^2 \times 0.04^2} = \pm 4.1 \text{ sq ft}$$

1-17. The Reduction of Errors From Art. 1-15 it is evident that the effect of the systematic error of the thermal expansion of a steel tape may be reduced by applying a computed correction to the observed value. Later it will be found that other kinds of systematic errors may be reduced by proper field procedures. For example, certain systematic errors are reduced by "reversing" the instrument between sights. Hence, there are two methods for reducing the effects of systematic errors: (1) by applying computed corrections to the field observations, and (2) by using approved field methods.

It should be noted, however, that neither of these methods eliminates entirely the effects of systematic errors. They may be reduced to such size that they become negligible and may be disregarded in the final result, but they are never completely eliminated.

Accidental errors are variable in magnitude and of unknown sign, so that they are not subject to computed corrections. Their effects are materially reduced by using good instrumental equipment and approved field procedures. Thus, in the example of Art. 1-15, the calculated value of the total accidental error due to setting pins is ± 0.07 ft. This is the estimated error of the result only and has the double sign, plus or minus. It cannot, therefore, be applied as a correction. If this calculated error is not sufficiently small, it can be reduced only by more careful field work.

1-18. Professional Surveying Practice It would be both unfortunate and incorrect to conclude that the serious study of any

surveying textbook would be in itself sufficient to qualify the student to engage in the professional practice of surveying. The design, execution, and administration of extensive surveying and mapping programs and the demarcation of important land boundary lines requires both a specialized technical education of college level and extensive experience of considerable responsibility. On the other hand, the performance of subordinate tasks, such as measuring angles with a transit, can be regarded as a subprofessional activity for which a technician with limited scholastic training would be considered amply qualified.

It is emphasized then that while there are many kinds of surveyors there is a fairly clear distinction between those who are professional engineer-surveyors and those who are technicians. This basic difference is recognized by Civil Service commissions in the classification of surveying positions and by the membership qualification standards of such organizations as the American Society of Civil Engineers.

The young man who desires to seek a professional career in surveying would be well advised to obtain a sound college-level technical education in some branch of engineering such as civil engineering and to secure specialized graduate training in several branches of surveying and the associated disciplines. Subsequent employment with a federal surveying agency, a consulting engineering firm, or a reputable registered land surveyor would provide the necessary background of experience needed to qualify for registration as a professional engineer or surveyor.

Technical organizations whose activities and publications are of particular interest to professional surveyors are the American Congress on Surveying and Mapping, the American Society of Photogrammetry, and the Surveying and Mapping Division of the American Society of Civil Engineers.

CHAPTER 2

MEASUREMENT OF DISTANCE

2-1. Remarks The instruments used in the measurement of distance, except by the stadia method, are of simple construction; and to the layman who observes it, the process of measuring appears simple. Consequently, the notion is generally held that anyone, regardless of his ignorance or inexperience in the work, is qualified to serve as a tapeman on a survey. It is true that under many conditions a person of average intelligence can be given a few simple instructions and serve acceptably on the survey. Yet it must be said that, in most work, the proper execution of the distance measurements is accompanied with more and greater difficulties than are those of angles. The engineer's transit, quite universally used for the measurement of angles, is an instrument of precision such that, with a little experience, the novice can measure the angles of a survey with greater relative accuracy and ease than the tapeman can measure the distances. Furthermore, mistakes are usually more frequent and more difficult to detect in the distance than in the angle measurements. Therefore, faulty results of surveying work are more commonly the result of the taping than the measurement of angles.

The transitman is usually given the responsibility of directing the work of the survey party in the field, and he has the important duty of keeping the notes; hence, he usually is a man of broader knowledge and experience than the tapemen. Insofar as the execution of manual duties is concerned, however, the experienced engineer in charge of a survey is as much concerned with the qualifications of his tapemen (especially the head tapeman) as he is of his transitman.

The beginner is likely to be so impressed with the more complicated transit and level instruments that he takes a superficial interest in the taping work. It is hoped that these remarks will dispel this attitude and that he will give his most serious and careful attention to the measurement of distance.

2-2. Units of Length The basic units of length used within the United States are the foot and the meter. The foot is of Anglo Saxon origin and is quite universally used in English-speaking countries. The meter is of French origin and has become the adopted unit for international and scientific usage. Its use in the United States in surveying work is limited practically to the precise measurements of geodetic surveys.

The decimal system of linear measurements, based on the foot unit, has been adopted for practically all surveying work. In engineering practice, the decimal system is extended to measurements of less than one foot, the foot unit being subdivided into tenths and hundredths. The building trades still use the older English units in which the foot is divided into inches, quarter-inches, etc.; and hence, architects and engineers frequently have to convert units from one system to the other.

The rod (sometimes called a perch, or a pole) is a unit of $16\frac{1}{2}$ ft that has considerable use in land measure.

The unit of land measure is the acre, which has been standardized at 1/8 mile in length and 1/80 mile in width. It is, therefore, 660 ft long and 66 ft wide. So it was, by reason of these dimensions, that the Englishman, Gunther, made use of a 66-ft chain. This made the acre 10 chains long and 1 chain wide, or 10 sq chains in area, and so reduced land measure to a decimal system.

The equivalents of the different units are as follows:

1 mile = 5280 ft = 1760 yards = 320 rods = 80 chains.
1 chain = 66 ft = 4 rods.
1 meter = 39.37 in. = 3.2808 ft = 1.0936 yd.
1 vara = 33 in. (California) = $33\frac{1}{3}$ in. (Texas).
1 acre = 43,560 sq ft = 10 sq chains.

2-3. Standards of Length It may be desirable to clarify the distinction between a unit of length and a standard of length.

A *unit*, whether of length or otherwise, is a value or quantity in terms of which other values or quantities are expressed. In general,

it is fixed by definition and is independent of such physical conditions as temperature.

A *standard* can be defined as the physical embodiment of a unit. Generally, it is not independent of physical conditions and it is the true embodiment of the unit only under certain specific conditions.

The primary standard of length in the United States is the distance between two engraved lines on a platinum-iridium bar of x-shaped cross section. The length of this bar, which is deposited at the National Bureau of Standards in Washington, D.C., is known in terms of the International Prototype Meter at the International Bureau of Weights and Measures near Paris, France. By act of Congress the relationship between the foot and the meter was effectively defined by the following legal relation:

$$1 \text{ U.S. yard} = \frac{3600}{3937} \text{ meter}$$

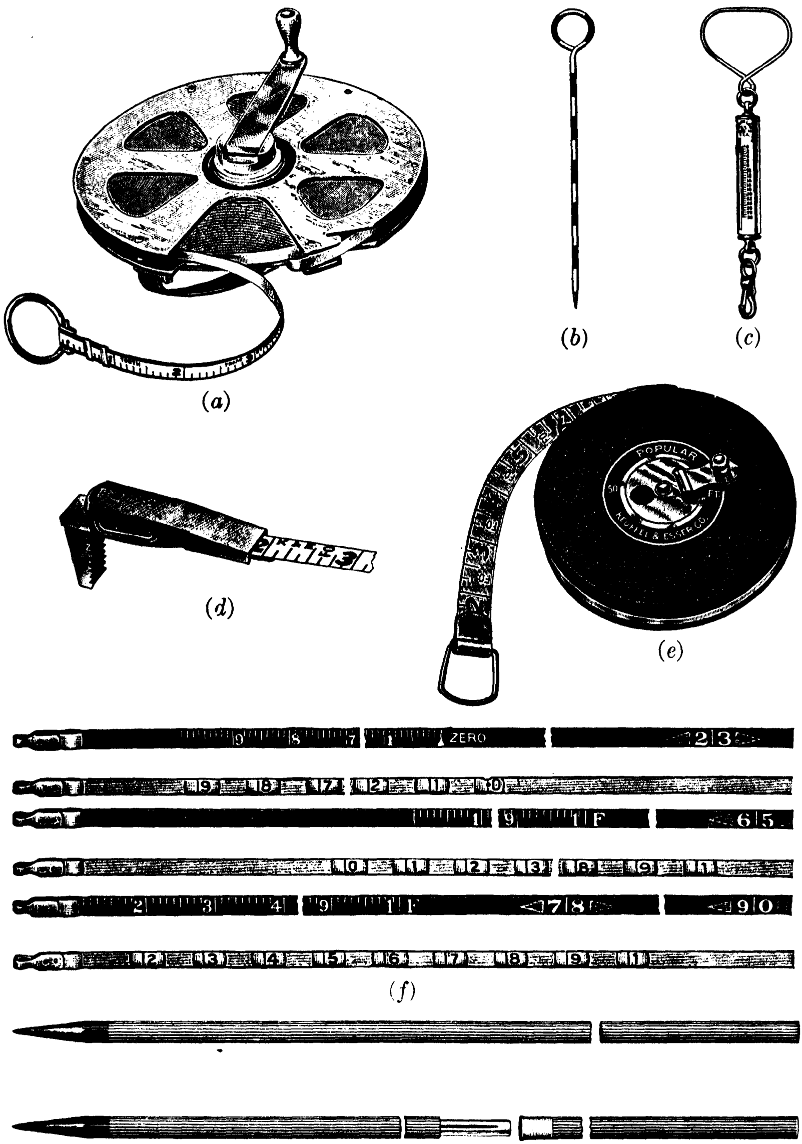
or

$$39.37 \text{ in.} = 1 \text{ meter}$$

In 1960 the International Meter was officially defined in a supplementary manner as 1,650,763.73 wave lengths of the orange-red light of krypton 86, a rare gas extracted from the atmosphere. The U.S. inch thus becomes equal to 41,929.399 wave lengths of the krypton light.

2-4. Instruments Illustrations of the more common instruments used in measuring distances are shown in Fig. 2-1. The steel tape is now used for most engineering measurements of distance. It is 66 ft long for government land surveys, or 100 ft long for general engineering purposes. Longer tapes consisting of full multiples of either kind are sometimes used. Formerly, a steel chain, either 66 ft or 100 ft long and composed of 100 links, was used, but this chain is now obsolete. Examples of the steel tape are shown at (F). It is usually carried in a looped circle without reel, but any tape which is not often used (and especially any standardized tape) is better kept on a reel as shown at (a).

The common tape is graduated by one of two means—either the numbers are stamped on soft (babbit) metal previously run on the tape at the foot divisions, or they are etched into the metal of the tape. The former is less expensive but somewhat heavier than the etched tape. However, the soft metal lugs are subject to wear



Courtesy of Keuffel and Esser Co.

Fig. 2-1. Measuring Instruments. (a) Steel Tape (100 ft) on Reel; (b) Steel Marking Pin; (c) Tension Handle; (d) Clamping Handle; (e) Woven Tape; (f) End Arrangements for Steel Tapes; (g) Steel Range Poles.

as the tape is dragged on the ground, thus rendering the numbers illegible. Consequently, the etched tape is the more satisfactory and economical of the two, especially if it is carefully handled so as not to be broken in use. The tape is marked at every foot throughout, and the foot-length at each end is generally subdivided into tenths of a foot. Tapes of invar metal (a nickel-steel alloy) are used for the most accurate measurements. Their principal characteristic is a very low coefficient of expansion, usually about 1/30 of that for a steel tape.

At (*b*) is depicted a *steel pin* used for marking the ends of the tape when measuring over ground where pins can be stuck in the surface; (*c*) is a *spring balance* used to apply tension to the tape when careful measurements are being taken; (*d*) is a *clamp handle* with which to take hold of the tape at any place along its length; (*e*) is a 50-ft *metallic tape*, used where many relatively short dimensions of ordinary precision are to be measured; (*g*) are *range poles* used by the tapemen to keep the tape on the line of measurement.

The range poles are occasionally of wood, 8 ft long, and shod with a steel point. They are more commonly made of steel tubing. They are painted in alternate 1-ft bands of red and white.

Metallic tapes are so-called because brass or copper threads are woven into the fabric to prevent undue stretching. However, such tapes are easily stretched in use, especially when wet, and should never be used for accurate measurements.

2-5. Care of Equipment The principal source of injury to the steel tape is that of breakage. The tape becomes looped while lying on the ground. Then when it is stretched into use, unless due care is exercised, it is kinked or broken. Accordingly, when in use, the tape, insofar as possible, should be kept free of twists or loops, and tension should never be applied without careful attention to this source of danger.

Steel tapes should be wiped dry after exposure to moisture to prevent rust. This is especially important with etched tapes.

Range poles should not be given any rough usage that would blunt the point or break the pole itself.

2-6. Pacing For many subordinate measurements of distance, pacing is sufficiently accurate and most expeditious. The length of

a pace varies, of course, with different persons and with the rate of speed. Sometimes, the yard pace is assumed but for many individuals it is difficult to maintain this pace uniformly and the results are not satisfactory, especially for long distances. Accordingly, the best practice is to use the natural pace and an average gait. In any case, the number of paces per 100 ft should be carefully determined. This may be done by laying the tape out on even ground and pacing its length at least four times, preferably more, estimating the last pace to the nearest half-unit, and maintaining a uniform average gait throughout. The length of pace may also be determined by pacing longer distances of known length, and having been thus determined, it may be applied proportionately to other distances.

2-7. The Stadia Method A most convenient method of measuring distances is that provided by the engineer's telescope with crosswires suitably arranged. It is called *the stadia method,* and, since it requires some understanding of the telescope, it will be described in Chapter 9.

2-8. Taping Over Level Ground The process of measuring with the steel tape is called "taping" and, assuming the ground to be level and open, the party consists of a head and rear tapeman. The equipment includes a 100-ft steel tape, a set of 11 taping pins, and one or two range poles.

Having arrived at the initial point, the head tapeman goes forward, if necessary, and sets the range pole for lining-in purposes. Meanwhile, the rear tapeman undoes the tape, laying it out in the general direction of the line to be measured, being careful that it is not looped or unduly twisted. The head tapeman then takes the *zero* end of the tape, hands one pin to the rear tapeman and moves forward along the line. When the 100-ft end of the tape comes up even with the point of beginning, the rear tapeman calls out "halt." At this signal the head tapeman halts and quickly places himself on line with the aid of right or left signals from the rear tapeman.

As soon as the tape has been placed on line, the rear tapeman holds his end of the tape exactly even with the initial point. The head tapeman takes his position just to the left of the line (not on the line), kneels, and applies tension (about 15 lb) to the tape with his left arm bearing against his leg. His right hand is then free to place the pin on line and at the zero mark of the tape. The pin

may be set vertically, but more often it is given a slant at right angles to the tape, by which it can be placed more conveniently and accurately in position.

When the head tapeman sets his pin, he should be assured that the rear tapeman is holding his end of the tape precisely on the mark, and the rear tapeman must not pull his pin until the head tapeman has finished setting his pin. Hence, before setting his pin, the head tapeman waits for the signal "right here" from the rear tapeman. As soon as he has set the pin he also calls out "right here," which is the signal for the rear tapeman to pull his pin. It is important that the signals be carefully observed.

At the initial point, marked, let us say, by a transit stake, the rear tapeman holds one pin and the head tapeman begins with ten pins on his ring. As soon as the head tapeman sets his first pin, the pin which the rear tapeman holds indicates the fact that one tape length has been measured. When the next pin is set, the rear tapeman pulls his pin and he now has two pins, indicating that two tape lengths have been measured. Accordingly, the number of pins which the rear tapeman holds in his hand, not counting the pin set in the ground, indicates the number of full tape lengths that have been measured. When the head tapeman sets his tenth or last pin, he calls out "tally." The rear tapeman now has ten pins which he brings forward, and the taping proceeds. Thus, the number of tallies indicates the number of thousands of feet which have been measured.

If the terminus of the line being measured is a previously fixed point, the last measurement will be a fractional tape length. It is here that mistakes in taping most frequently occur, and care must be exercised that the procedure is systematic and always the same, to avoid confusion.

When the end of the line is reached, the head tapeman halts and the rear tapeman comes up to the last pin set. The tape is quickly adjusted so that an even foot mark is opposite the pin, and the terminus falls within the end foot length which is subdivided into tenths. Tension is applied, the head tapeman observes the number of tenths, estimating hundredths if necessary, which extend beyond the terminus, and the rear tapeman observes the number of the foot mark he is holding at the pin. The number which the head tapeman observes is subtracted from the number the rear tapeman reads to obtain the measured fractional distance. For example, the head tapeman observes 0.28 ft as that part of the tape which extends be-

yond the terminus, and the rear tapeman observes his foot mark to be 35 ft. The head tapeman then calls out "Cut twenty-eight hundredths," the rear tapeman calls out "Thirty-five," and they both make the subtraction mentally, and check each other on the result, 34.72 ft. With some tapes, the fractional distance can be read directly.

If the rear tapeman holds 7 pins in his hand, not counting the one in the ground, the total distance is 734.72 ft.

If the taping is done on a hard surface, such as a sidewalk, steel rail, or pavement, the position of the end of the tape is marked with a colored lumber crayon, called *keel*. In this case the number of the tape length is recorded beside the mark as a means of keeping the count of tape lengths measured. To avoid mistakes, the rear tapeman calls out the number of his mark just before the head tapeman records the next number.

2-9. Taping Over Sloping or Uneven Ground In plane surveying all distances measured on the earth's surface are taken to be the projections of these distances on a horizontal plane. In other words, horizontal distances only are considered in final results. These are obtained by one of two methods—either the tape is held horizontally while all measurements are taken, or the tape is held on the ground slope and a correction is applied to the slope measurement.

In Fig. 2-2 consider the two points A and B located on a slope, and between which the horizontal distance is 500 ft. Let h = a horizontal distance of 100 ft; v the vertical fall of the slope for each tape length; s = the slope distance corresponding to the horizontal 100-ft distance; and C_g = the correction, or difference between the horizontal and the slope distance.

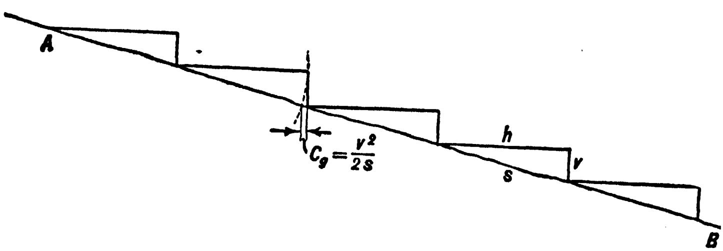


FIG. 2-2. Surveying a Slope.

Evidently, the correct horizontal distance between A and B will be measured if the tape is held horizontally for each tape length and if the position of the end of the tape is projected vertically to the ground by means of a plumb line; also, if the measurement should be taken on the slope and the proper corrections applied. Each method will now be described and the merits of the two methods will be discussed presently.

First Method; the Tape Level.—When, on sloping ground, the tape is held horizontally, each tapeman should carry a plumb bob, and possibly a hand level, for estimating the steepness of slopes.

If the slope is downhill, the head tapeman estimates the vertical distance v, holds his end of the tape at that elevation above the ground, applies the tension, and by means of the plumb bob transfers the position of the end of the tape to the ground, where a pin is set or, on a hard surface, a keel mark is made. If accurate work is being done, the tape is stretched a second time and the mean of the two measurements taken.

It is not convenient, or good practice, to hold the tape more than 5 ft above the ground, and hence, if the slope is more than 5 ft per tape length it is necessary to "break" tape, as the process is called. In this case the head tapeman pulls the tape forward until the rear end comes up to the rear tapeman. He then goes back until he reaches a point where the vertical distance v is not more than 5 ft. Here, at some full foot mark, he plumbs down to the ground. The rear tapeman then comes up, holds the foot mark at the ground point, and the head tapeman goes forward until another point is found for which v is approximately 5 ft, where he again plumbs down and fixes a new ground point. This process is repeated until the full tape length has been measured. It may be noted that it is immaterial what foot marks on the tape are used, and no record of them is kept.

The head tapeman finds that it is less work to use a different method, by which he goes forward only until he reaches the first point where the tape is broken. The rear tapeman reads the foot mark and they proceed again. The full tape length is then found by adding together the separate fractional lengths. This practice is strongly to be condemned, because it too often results in mistakes being made in reading the tape and in adding the lengths together. On important surveys the method is prohibited.

On less accurate surveys, a range pole may be used to plumb from the end of the tape to the ground. In going uphill, of course,

the rear tapeman must hold his end of the tape above the ground and its position, likewise, is found by plumbing from the ground point.

In taping over rough ground where there is dense vegetation such as cornstalks, weeds or underbrush, it may be difficult or impracticable to hold the tape on the ground even though the slope is negligible. In this case, a plumb line must be used at each end of the tape.

Second Method; Tape on the Ground.—Wherever the tape can conveniently be held on the ground, no matter how steep the slope, it should be done; because, as will be found when the sources of error are discussed, this method is more accurate and rapid than attempting to hold the tape horizontal and plumbing to the ground. The only difference between this method and that of taping on level ground is that a correction must be applied, the magnitude of which will now be considered.

In Fig. 2-2 it is evident that the value of the correction C_g is the difference between s and h; the hypotenuse and vertical leg of the right triangle whose sides are s, h, and v.

The ratio of the sides v/h is called the *gradient* of the slope and is usually expressed in per cent; i.e., the rise or fall in a distance of 100 ft. Thus, a 1% grade means one for which the vertical rise v is 1 ft for a horizontal distance of 100 ft. The gradient is sometimes expressed in degrees of arc, indicating the vertical angle between the horizontal and the slope, but this practice is not common in taping.

Evidently, the correction C_g is equal to the difference $s - h$, which can be found exactly from the right triangle as follows: $s^2 = h^2 + v^2$, or $s^2 - h^2 = v^2$, from which

$$(s - h)(s + h) = v^2$$

or

$$(s - h) = \frac{v^2}{s + h} \tag{2-1}$$

The usual condition for which the value of C_g is desired is that in which the value of v is known (measured in the field) and the slope distance is 100 ft; hence, h is unknown. In the right-hand member the ratio $v^2/(s + h)$ is usually a small number, and since s and h are nearly equal in magnitude, the error introduced will be small if s and h are assumed to be equal. With this assumption, the equation becomes

$$C_g = \frac{v^2}{2s} \text{ (approx.)} \tag{2-2}$$

This correction to reduce a slope measurement to the horizontal is always negative. Under certain circumstances, however, when it may be necessary to set points one full "station" or 100 ft apart (horizontally), the following different manner of applying the correction should be noted. As the tapemen proceed in the field, the head tapeman estimates the slope, either up or downhill, makes a mental calculation of the correction, and sets the pin, or makes his mark, at the calculated distance beyond the end of the tape. This establishes a distance whose horizontal projection is 100 ft and, therefore, the same horizontal distance that would have been measured if the ground were level. By this procedure no tabulation in the notebook or subsequent corrections are necessary.

It may be added that a third method of measuring on slopes is that by which the distance is measured on the slope, the inclination of which is found on the vertical arc of a transit, or a clinometer. Obviously then, $h = s \cos \alpha$, where α is the measured angle of inclination. This relation yields exact results and is easy to apply where the vertical angle can be measured conveniently.

2-10. Perpendiculars and Angles It is frequently desirable to erect perpendiculars and sometimes to measure angles with a steel tape. In Fig. 2-3a it is evident that line CD, perpendicular to AB,

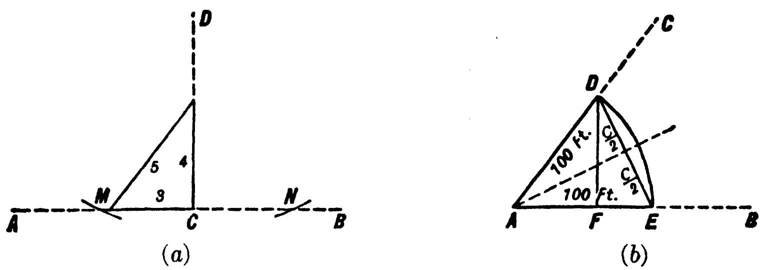


FIG. 2-3. Perpendicular and Angles with a Tape.

can be erected at point C by use of the 3:4:5 ratio. The lengths of the sides may be any common multiples of this ratio and will be determined by the length of line CD, which is to be established, and by the accuracy desired.

It is also evident that perpendicular DC can be dropped from point D to line AB by the common method of swinging an arc to establish points M and N. The foot of the perpendicular at C is midway between them. Such a perpendicular is called a "swing offset."

The angle CAB, Fig. 2-3b, can be readily measured by swinging an arc with a radius of 100 ft to establish points D and E. The chord distance $C = DE$ is then measured and the value of the angle is given by the relation,

$$\sin \frac{CAB}{2} = \frac{C}{2 \times 100}$$

Obviously other trigonometrical relations may also be used. Thus, if a perpendicular were dropped from D upon line AB, then

$$\tan CAB = \frac{DF}{AF}.$$

2-11. Errors in Taping The principal sources of error which affect measurement with the steel tape and their principal characteristics are as follows:

(a) *Incorrect Length.* The length of a steel tape varies with the conditions as to temperature, pull, and sag. Therefore, a tape is said to have its correct length only under given conditions. The standards for weights and measures in the United States are fixed by the Bureau of Standards at Washington, and a typical certificate of comparison for a steel tape is shown in Fig. 2-4.

A steel tape purchased from reputable makers usually will have a length not different from the standard by more than 0.01 ft, but a tape that has been in use may have become kinked or patched so that its length has been appreciably altered. Such tapes are frequently as much as 0.02 ft and sometimes 0.1 ft in error. Hence, no tape that has not been standardized should be used on important work. If no standard is available in the vicinity, the Bureau of Standards will make the comparison for a small charge.

The effect of this source of error is greatly reduced by applying the proper corrections, but it is not entirely eliminated, because the comparison with the standard is not exact.

The matter of applying corrections due to an incorrect length of tape should receive careful attention. While measuring with a 100-ft tape, its length is assumed to be 100 ft exactly. Hence, the measured

NBS-579 A
(Rev. 12-15-[illegible])

UNITED STATES DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS
WASHINGTON 25, D.C.

National Bureau of Standards

Certificate

100-Foot
Steel Tape

Maker: Keuffel & Esser Co. Submitted by NBS No. 12073

University of Illinois
College of Engineering
Department of Civil Engineering
Urbana, Illinois

This tape has been compared with the standards of the United States, and the intervals indicated have the following lengths at 68° Fahrenheit (20° centigrade) under the conditions given below:

Supported on a horizontal flat surface

Tension (pounds)	Interval (feet)	Length (feet)
10	0 to 100	100.001
20	0 to 100	100.008

Supported as indicated below

Tension (pounds)	Points of Support (feet)	Interval (feet)	Length (feet)
20	0, 50 and 100	0 to 100	100.000
20	0 and 100	0 to 100	99.976

See Notes 3(a) and 5(b) on the reverse side of this certificate.

For the Director,

B L Page

B. L. Page
Chief, Length Section
Metrology Division

Test No. 2.4/G-28869

FIG. 2-4. Steel Tape Certificate.

length of a line, i.e., the value observed and recorded in the notebook, is that for a tape exactly 100 ft long. Then, if the actual length of the tape, when compared with a standard, is found to be 100.02 ft, the true length is 100.02, although the "recorded" length is 100.00 ft. Hence, if the tape is too long, the correction must be added to the recorded length.

For example, if a distance is measured with the tape just mentioned and found to be 705.76 ft, the resultant error would be $7.05 \times 0.02 = 0.14$ ft, and the corrected length, therefore, would be $705.76 + 0.14 = 705.90$ ft.

Likewise, if the tape is too short, the correction must be subtracted from the recorded length.

It should be noted that these corrections are made when a distance is being measured between existing end marks. If two end marks are to be established on the ground at a previously determined distance, then the signs of the corrections will be reversed. For example, if, in staking out a city subdivision, it is necessary to set two iron pins exactly 600 ft apart, and if the true length of the tape is 100.02 ft, the measured distance with this tape will be $600.00 - 0.12$ ft $=$ 599.88 ft.

(b) *Temperature.* The length of a steel tape varies with the temperature, and this condition is the source of a systematic error. The error is given by the relation

$$E_t = CL(T_l - T_o) \tag{2-3}$$

in which C is the coefficient of expansion of steel, L the length of the tape, T_o the temperature at which the tape has its standard length, and T_l the temperature when the tape is in use.

The commonly accepted value of C for steel tapes is 0.0000065 per degree Fahrenheit. Hence, if $T_o = 60°$, the error in a 100 ft tape for a temperature of 75° is found to be

$$E_t = 0.0000065 \times 100(75 - 60) = 0.01 \text{ ft}$$

Thus it is noted that a 100-ft tape will have its length changed 0.01 ft for each change of 15°F in the temperature. For small ranges of temperature and on ordinary work, this error may not be important, but the inexperienced engineer is apt to underestimate the importance of this source of error when even the ordinary temperatures of winter and summer measurements are encountered. For example, a summer temperature of the tape of 100°F and a winter

temperature of 25°F are not uncommon. This difference in temperature of 75°F causes a change in the length of the tape of 0.05 ft, which makes a discrepancy of 2.6 ft in a mile. This error is greater than that permitted from all sources combined, on many surveys, and yet it is frequently disregarded entirely.

On careful work, thermometers are used to determine the temperature of the tape, and on precise work, tapes of invar metal having a very low coefficient of expansion (about 1/30 of that of steel) are used.

(c) *Slope.* The nature and magnitude of this source of error have been discussed in Art. 2-9. It may be added that the error is systematic when no corrections are applied.

(d) *Alinement.* The effect of inaccuracy in keeping the tape on line is the same in nature and magnitude as that due to slope. However, it is much more easily controlled than is the effect of slope, and the resulting errors are usually much smaller. Of course, the tape should be kept on line, within proper limits, but when it is remembered that a 1% slope causes the same error as 1 ft error in alinement, it shows considerable ignorance on the part of the tapemen if they use an undue amount of time to "line in" the tape to the nearest 0.1 ft and disregard entirely a slope of perhaps 3 or 4%.

(e) *Sag.* Whenever a tape is held off the ground it sags, and unless it has been standardized under the same conditions, the effect is that of shortening the tape and thus introducing a systematic error.

The magnitude of the error depends on the weight, the unsupported length, and the pull on the tape. It is computed by the equation

$$E_c = \frac{W^2L}{24P^2} \tag{2-4}$$

in which W = the weight (in pounds) of tape between supports,
L = the interval between supports,
P = the tension (in pounds) on the tape.

Example: A 100-ft steel tape weighs 2 lb and is held supported at the ends only with a pull of 12 lb. Find the error due to sag.

$$E_c = \frac{2^2 \times 100}{24 \times 12^2} = -0.11 \text{ ft}$$

If the tension were increased to 20 lb, the shortening is reduced from 0.11 ft to 0.04 ft, which shows the desirability of using a higher

tension on the tape when unsupported, and also the fact that the error in any case is considerable. The error is, of course, reduced by applying corrections, but they are not readily determined; therefore it is better practice, when conditions permit, to avoid the effects of sag by taping on the ground.

(f) *Setting Pins.* The errors due to setting pins or marking the tape lengths are accidental in kind and their magnitudes depend on the care used.

(g) *Tension.* Since a steel tape is elastic to a small extent, its length is changed by variations in the tension applied. This change is not to be confused with the effect on the sag of a tape due to changes in tension; but rather, when the tape is unsupported, it is that change in length due to a change in tension only.

It can be calculated from the expression

$$E_p = \frac{(P_1 - P_0)L}{AE} \tag{2-5}$$

in which E_p = the elongation of the tape of length, L, in feet
P_1 = the applied tension, in pounds
P_0 = the standard tension, in pounds
A = the cross-sectional area of the tape, in square inches
E = the modulus of elasticity of the tape material (for steel 29,000,000) in pounds per square inch

It will be sufficient here to state that an ordinary 100-ft steel tape will stretch about 0.01 ft for a change of 15 lb in tension. Since there is little occasion for any but slight changes in this source of error, and since the tension may be assumed to vary either above or below the standard tension, it is an accidental error and may be disregarded in all but precise measurements. The "standard" tension, ordinarily used, should be about 15 lb.

(h) *Wind.* If a tape is stretched unsupported and a strong wind is blowing, the center of the tape will be carried to one side of the line joining the two ends. This condition causes an effect similar to, but usually much less than, sag.

A review of the previous discussion shows that (1) most of the errors in taping are systematic, and accordingly they vary nearly with the distance measured; (2) the magnitudes of the errors can be greatly reduced by applying simple corrections; and (3) the effects of temperature and slope are likely to receive too little, and alinement too much, consideration.

2-12. Mistakes Some of the common mistakes made in taping and recording are listed below:

1. *Omitted Tape Length.* The serious mistake of omitting or adding a tape length is to be prevented by orderly procedure and careful attention to the work. The manner of checking is stated in the following article.

2. *Misreading the Tape.* A frequent mistake is that of misreading the tape, as when 6 is read for an inverted 9. Thus 86 is read for 89, or vice versa. Also, when the numbers on a tape become worn, an 8 may be read as a 3, etc. Mistakes of this kind are prevented if the tapemen will form the habit of looking at the number of the next adjacent mark before and after the reading has been made.

3. *Calling and Recording Numbers.* Numbers are easily reversed or misunderstood when they are called out to be recorded. The zero digit and the decimal point are most likely to cause mistakes. Thus the number 40.4 should be called as "forty, point four." Otherwise, it may be misunderstood as "forty four" and recorded as 44.0. The recorder should always repeat such numbers as are called to him, before recording them.

4. *One-Foot Mistakes.* It is easy to make a mistake of one foot when measuring a fractional tape length. Accordingly, care should be taken to use the method given in Art. 2-8 and to use the same method always.

5. *Mistaking the End Mark.* The end marks are differently arranged on the different tapes. Hence, the tapemen should assure themselves of the position of the end marks of each tape before it is used.

2-13. Checks In general, the field check that can be applied to the measurement of distance consists in a duplicate measurement. However, the engineer must be careful to remember that any discrepancies found between measurements made under similar conditions do not reveal any systematic errors. Thus duplicate measurements of a distance of one mile might show a discrepancy of 1 ft, but if the tape were 0.1 ft too long there would be an error of 5 ft from that source alone, which would not be indicated by the discrepancy. Since most errors in taping are systematic, under ordinary conditions, it is important that the engineer be not deceived with regard to the apparent precision indicated by small discrepancies between duplicate measurements that, in reality, are seriously in

error. Careful attention must be given to the various sources of error such that repeated checks under various conditions will show results within the desired accuracy.

Duplicate measurements, however, do serve the important purpose of detecting mistakes and are frequently made. The most serious mistakes are those of omitted tape lengths in taping, or misreadings of the rod in stadia work. The duplicate measurements necessary for checking need not be as accurate as the original. Hence, a distance which has been measured by the stadia, may be checked by pacing; or, a distance found by taping may be checked by stadia.

2-14. Accuracy The variety of conditions the engineer meets in the field prevents the making of any definite statement as to the accuracy that may be expected by the use of the different methods discussed here, yet it is desirable that anyone engaged in a survey should have a knowledge of the approximate degree of accuracy that the different methods should yield. Accordingly, the risk will be taken of stating a few values that the experienced engineer will know do not apply to all conditions and that he should use every opportunity to verify or to modify according to his later experience.

Because, for ordinary work, the principal errors in measuring distances are systematic, the resultant error is nearly proportional to the distance measured, and the accuracy of results is expressed by the ratio of the error to the distance. Thus, the ratio 1/3000 expresses that accuracy in which the error is 1 part to 3000 parts of distance. For comparative purposes, such ratios are always expressed with the numerator as unity, and with the denominator in round numbers only. Thus, an error of 3.4 ft in a distance of 4346.8 ft would be expressed as 1/1300.

For average conditions in the open country, good work in pacing, stadia, and taping is represented by the ratios of 1/50, 1/500, and 1/5000, respectively. For fair results the accuracy may be taken as about half, or 1/25, 1/250, and 1/2500, respectively. Rough taping may be taken as 1/1000. As regards taping, if it is done on graded roadways, the ratios mentioned above are easily doubled.

2-15. Specifications To indicate more definitely what is meant by each of the three grades of taping mentioned above, specifications are given below which may be expected to yield the desired accuracy for the assumed conditions.

Conditions. It is assumed that the average conditions as to weather, equipment, and personnel obtain; that the line is measured across country where the ground is rolling or hilly, partly open and partly wooded so that some of the taping is done with the tape on the ground and some with it unsupported; that the tape has its standard length at 68°F supported throughout and under a tension of 15 lb.

Three ratios of accuracy are specified—1/5000, 1/2500, and 1/1000. An accuracy of 1/5000 signifies that the total linear error due to taping a course may be expected to be not greater than 1 ft in 5000 ft.

1/5000. (a) *Length of Tape.* The length of the tape should be determined within ±0.01 ft and the proper correction applied.

(b) *Temperature.* The temperature of the tape should be determined within ±10°F and the proper corrections applied.

(c) *Slope.* All slopes should be estimated within 2% and the proper corrections applied.

(d) *Alinement.* The alinement should be correct within 1 ft.

(e) *Sag.* When unsupported, the tension on the tape should be 15 lb within ±3 lb.

(f) *Tension.* Disregard variation, but use a tension of 12 lb when taping on the ground.

(g) *Setting Pins.* End of tape to be marked within ±0.03 ft.

* * *

1/2500. (a) *Length of Tape.* Length of tape to be determined within ±0.01 ft, and if necessary, the proper corrections to be applied.

(b) *Temperature.* Disregard ordinary temperatures but apply corrections for those above 90° or below 30°F.

(c) *Slope.* Disregard slopes of 1 or 2%, but apply corrections or break tape for others; the errors of estimation not to exceed 2%.

(d) *Alinement.* Alinement to be correct within 1 ft.

(e) *Sag.* Disregard variations, but use a tension of 15 lb.

(f) *Tension.* Use a tension of 12 lb.

(g) *Setting Pins.* End of tape to be marked within ±0.05 ft.

* * *

1/1000. (a) *Length of Tape.* Tape to be of average quality in good condition, but not standardized.

(b) *Temperature.* Disregard.

(c) *Slope.* Disregard slopes up to 5% and break tape on others.

(d) *Alinement.* Ordinary care to be exercised.

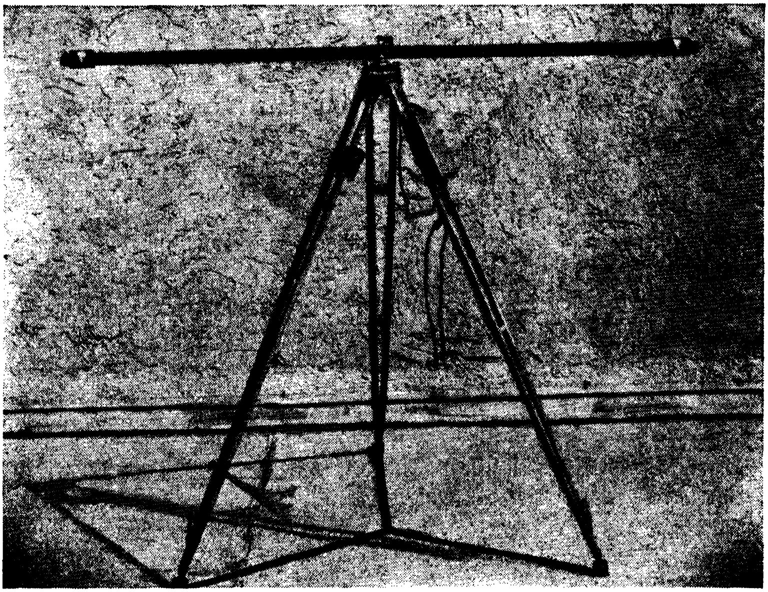
(e) *Sag.* Disregard variations but use 15 lb tension.

(f) *Tension.* Use a tension of 12 to 15 lb.

(g) *Setting Pins.* Use ordinary care.

2-16. The Subtense Method In recent years special methods for indirectly making distance measurements have been developed. Included are the *subtense method* and certain electronic length measuring procedures.

In the subtense method distances are obtained by observation of the horizontal angle subtended by targets fixed at a known distance apart on a horizontal bar. The subtense bar is usually made of invar, and the distance between the targets is accurately established at some standard length. In the subtense bar shown in Fig. 2-5, this length is 2 meters.



Courtesy of Wild Heerbrugg Instruments, Inc.

FIG. 2-5. Subtense Bar.

The procedure for determining the distance between two points consists merely of setting up the subtense bar at the distant station, aligning it at right angles to the course by means of an attached sighting device, and then measuring the horizontal angle subtended by the distance between the two targets.

Fig. 2-6 shows the principle of the subtense method. It is evident that the horizontal distance is given by the expression

$$D = \frac{1}{2} S \cot \frac{\alpha}{2} \tag{2-6}$$

Tables furnished by the maker of subtense equipment simplify the determination of the distance. The quality of subtense bar determinations of distance obviously depends upon the accuracy with which the angle, α, is measured.

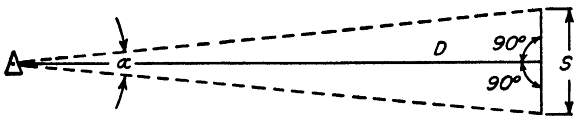


FIG. 2-6. Principle of Subtense Method.

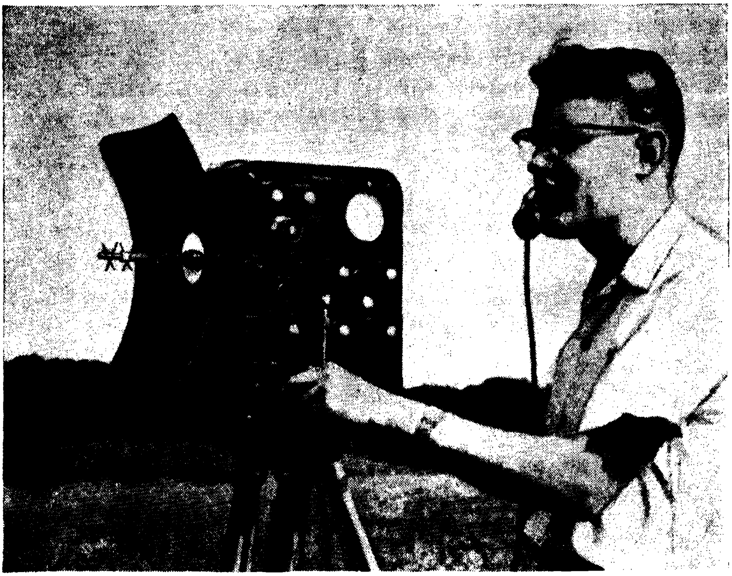
However, even with the highest quality angle measuring instruments, it is difficult to obtain a relative accuracy of 1/5000 unless the sight distances are limited to 500 or 600 ft.

Despite its limitations, the subtense method has several advantages. It can be used to measure distances over very rough terrain, across gullies, and wide streams. Since the horizontal angle subtended by the subtense bar is independent of the inclination of the line of sight, the horizontal distance is obtained directly and no slope correction is required. The accuracy of traversing long courses may be increased by subdividing the courses into small sections. Also, since most of the errors associated with subtense work are accidental, they tend to compensate each other and make possible the attainment of higher accuracies on longer lines.

2-17. Electronic Methods The application of electronic techniques to surveying is a direct outgrowth of military uses of radar in air navigation and precision bombing during World War II. Electronic surveying, as it is sometimes called, is primarily concerned with the rapid and accurate determination of distance on high quality surveys of considerable extent. The basic principle of the two types of apparatus which will be briefly described is that the time required for a radio or light wave to travel from one survey point to another is a function of the distance between the stations. Hence, if the interval of time between emission and reception of the signal is very carefully measured, the distance (slope) is directly provided by the product of the velocity and the elapsed time. In

effect, then, such devices are essentially very accurate time-interval measuring instruments.

The *tellurometer system,* Fig. 2-7, consists of two portable instruments which are mounted on ordinary tripods over the survey points



Courtesy of Tellurometer Inc.

FIG. 2-7. Tellurometer, Model MRA-2.

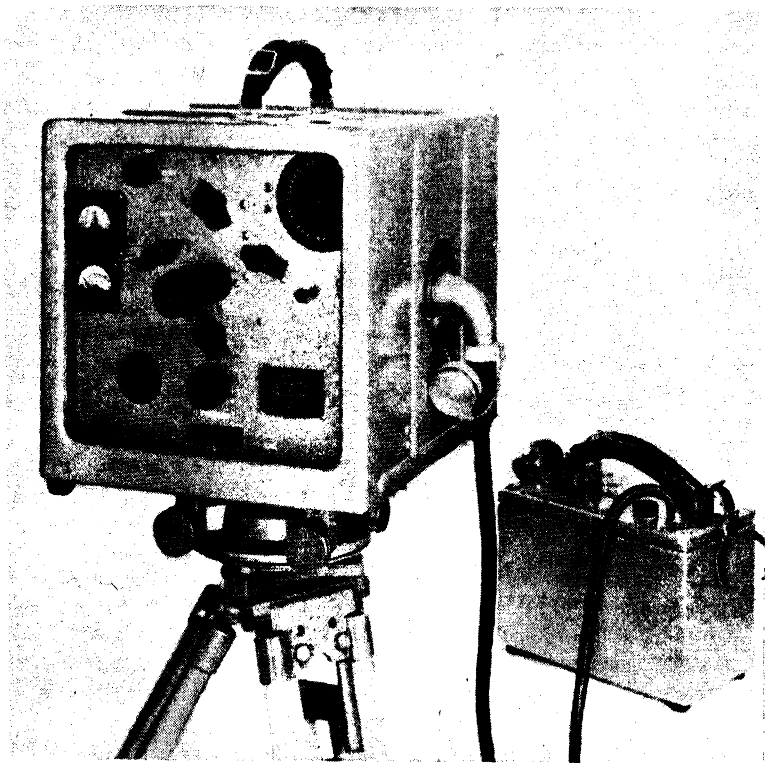
between which the distance is desired. One of the instruments transmits a series of radio waves toward the receiver set up over the other station. These impulses are run through the circuitry of the receiver and retransmitted to the sending unit. The time (in millimicro seconds) required for the impulse to travel the round trip is indicated on the panel of the transmitter and these readings can then be readily converted into distance.

The operation of the tellurometer system is rapid since the time required to take a measurement, from setting up the equipment to dismantling it, is only 30 minutes. It weighs only 28 lb and works as well in fog, rain, or darkness as in clear weather.

The accuracy of the tellurometer system has been well established at 1 part in 300,000 of the length ±2 in.

The *geodimeter,* Fig. 2-8, is an electro-optical device which pro-

jects a pulsating light beam to a reflector which returns the light to the instrument. By a special technique, a comparison is made between the transmitted and received light in order to ascertain the time required for the light pulses to make the round trip.



Courtesy of AGA Corp. of America

FIG. 2-8. Geodimeter, Model 4.

Three different geodimeter models are in use. Each instrument has its own specific area of application, although these overlap somewhat. The most portable (36 lb) geodimeter is the Model 4. Actual measuring time is only 10 minutes. It can be used to measure distances up to 1 mile under daylight conditions even though it is primarily adapted for use at night, when its range is 8 miles.

Its accuracy has been stated as 1 part in 200,000 of the distance ±0.5 in.

It should be noted that the inherent accuracy of the tellurometer system and the geodimeters is directly related to the correctness of the assumed velocity of light and radio waves *in vacuo*. The value officially adopted by the International Scientific Radio Union is 299,792.5 ±0.4 km/sec. However, some authorities believe this value to be uncertain by as much as 1-3 km/sec.

For both tellurometer and geodimeter measurements it is necessary to make temperature and pressure observations in order to calculate the actual velocity of the radio and light waves through the atmosphere.

Office Problems

Suggestion: It is important that the subjects of *significant figures* and *consistent accuracy* in computations (see Arts. 7-2 and 7-3) be well understood and applied to all computations based on standard quantities. Much unnecessary labor and resulting mistakes will be avoided in computing the corrections to taped distances if it is noted that (*a*) most of the corrections are small quantities which may be computed mentally or by slide rule, (*b*) the total corrections are more easily found if they are computed for one tape length and then multiplied by the number of tape lengths, and (*c*) the different corrections may be computed independently of each other.

2-1. A distance has been measured and found to be 2418.6 ft. Later the tape was standardized and found to be 99.97 ft long. Find the correct length of the line. *Ans.* 2417.9 ft.

2-2. A distance of 716.32 ft was measured on an average slope of 6% with the tape held on the ground. Find the correct horizontal distance. *Ans.* 715.03 ft.

2-3. A steel tape has been standardized and found to be 100.01 ft long at 68°F. A distance of 2641.7 ft has been measured with this tape when the average temperature was 88°F. The coefficient of expansion: 0.0000065 for 1°F. What is the correct length of the line? *Ans.* 2642.3 ft.

2-4. In staking out a city subdivision it is necessary to establish a horizontal distance of 634.28 ft. The tape is 0.02 ft short and is held on the ground which has a slope of 4%. What is the field measurement required? *Ans.* 634.91 ft.

2-5. A standardized tape has the following characteristics: Length: 100.02 ft at 68°F, supported throughout under a tension of 12 lb; weight: 2 lb; coefficient of expansion: 0.0000065 for 1°F. A distance of 3674.28 ft was measured with this tape on a 4% slope with the tape on the ground and at a temperature of 28°F. Compute the

corrections separately for length of tape, slope, and temperature, and find the corrected length of the line. *Ans.* 3671.11 ft.

2-6. For the standardized tape of Prob. 2-5, calculate the effect of sag for a tension of 15 lb (*a*) for the tape supported at the 0, and 100-ft marks, and (*b*) for the tape supported at the 0, 50-ft and 100-ft marks. *Ans.* (*a*) 0.074 ft. (*b*) 0.018 ft.

2-7. A distance has been measured and found to be 1317.6 ft. Later the tape was standardized and found to be 100.02 ft long. Find the correct length of the line.

2-8. A distance of 3715.4 ft was measured on an average slope of 4%, with the tape held on the ground. Find the correct horizontal distance for 100 ft. For the total distance.

2.9. A distance was measured at a summer temperature of 90°F and found to be 6325.7 ft. Later at a winter temperature of 30°F it was measured as 6328.6 ft. What part of this discrepancy was due to the change in temperature only? Coefficient of expansion: 0.0000065 for 1°F.

2-10. A mill building 80 ft × 150 ft is to be staked out with a 50-ft steel tape which is 0.01 ft too short. What field dimensions will be necessary?

2-11. A standardized tape has the following characteristics. Length: 99.98 ft at 62°F, supported throughout and under a tension of 12 lb. Weight: 2 lb. Coefficient of expansion: 0.0000065 for 1°F.

A distance has been measured in the field with this tape and found to be 5291.4 ft under the following conditions: Average slope: 3%; tape held on the ground; temperature: 40°F. Find the corrected length of the line. Express the total error as a ratio to the length.

2-12. For the tape of Prob. 2-11, suppose the same distance had been measured with the tape held horizontally and unsupported, with a tension of 12 lb. Calculate the effect of sag and compare with the effect of slope.

2-13. For the tape of Prob. 2-11, calculate the effect of sag for one tape length for the various tensions of 8, 10, 12, 15, and 20 lb, respectively.

2-14. A line *AB* cannot be measured directly because of obstructions on line. Accordingly, the two lines *AC* and *CB* were measured as 2100 ft and 1310.4 ft respectively. Point *C* was at a perpendicular distance of 30 ft from line *AB*. By the approximate formula of Art. 2-9, find the distance *AB*.

2-15. In measuring the fractional tape length at the end of a line, the slope distance is 30.20 ft, the difference in elevation for this distance being 6.0 ft. Find the horizontal distance by use of the approximate formula of Art. 2-9. Compare with the exact value.

Field Problem 2-1. Taping Over Level Ground

Procedure.—Begin at corner *A* of the field and tape each side sep-

arately both in a forward and backward direction. Use the procedure given in Art. 2-8 and record the results as shown in Fig. 2-9. Record the field temperature. Either before or after taping the field, compare the tape with a standard, to the nearest 0.01 ft, and record the tem-

	TAPING OVER LEVEL GROUND					Locker 32	May 19, 19 31
	Distance						R. Hansmeir, Tape
Course	Forw'd	Back	Mean.	Cor.	Correct Dist.		J. P. Joyce , "
A-B	178.20	178.16	178.18	+0.04	178.22		Temp. 80°F Cloudy
B-C	289.81	289.87	289.84	+0.07	289.91		
C-D	362.12	362.22	362.17	+0.08	362.25	Tape compared	with standard
D-E	311.03	311.05	311.04	+0.07	311.11	Tape U.S.B.S.	No. 3248 and found
E-A	222.16	222.12	222.14	+0.05	222.19	to be 100.01 ft	at 60°F supported
	1363.32	1363.42				throughout.	

Discrepancy between forward and back measurement = 0.1 ft $= \frac{1}{13000}$

FIG. 2-9. Field Notes for Taping.

perature. If there is as much as 10° difference between the field and the comparison temperatures, proper corrections should be applied to find the true length of each side. Special care should be taken to avoid mistakes in reading the fractional distance at the end of each course. The difference between the total forward and the total backward measurements is the discrepancy which should not exceed 1/5000.

Field Problem 2-2. Taping Over Sloping Ground For taping on a slope the following procedure is suggested: (1) Measure the course forward and back by the method of Art. 2-9, using a hand level to determine when the tape is held level. If the head and rear tapemen keep the same positions when taping forward and back, each man will get the same experience with the hand level and the plumb bob.

(2) Measure the same course by the second method of Art. 2-9, keeping the tape on the ground and using the hand level to determine the vertical height to use in computing the slope correction. By this method also, the course should be measured forward and back, the tapemen keeping the same positions throughout. The amount of the corrections can be computed mentally or, at first, a table of corrections, recorded in the back of the field book, can be used.

It is suggested that the specifications for an accuracy of 1/5000 be used, and the discrepancy between a forward and back measure-

ment should not exceed 1/2500. All slopes of 1% or less may be disregarded.

Field Problem 2-3. Taping an Obstructed Distance

Procedure.—There are three methods of taping an obstructed distance (see Fig. 6-9): (a) by equilateral triangle, (b) by rectangular or parallel offsets, and (c) by swing offset, Art. 6-17. Tape an assigned distance by one method and check by another. Special care should be taken when lining in all points.

CHAPTER 3

DETERMINATION OF ELEVATION

3-1. Remarks Information regarding the relative elevation of points on the earth's surface is necessary for all construction projects. Thus the ground and basement floor levels of buildings must be fixed with a proper relation to street and sewer elevations; all route projects such as street pavements, highways, railways, canals, drainage channels, etc., are built to fixed gradients for which purpose leveling operations are used continuously; the volumes of earthwork, of reservoirs, and of the flow of streams can be computed only by the use of information provided by leveling. These examples are mentioned only to indicate how prevalent and important leveling is in all engineering works of construction.

Nearly all informational surveys represent in some manner the relief of the ground surface, and, therefore, the elevations of many points are required. Since these surveys are of wide extent, the lines of levels are of great length; and to prevent the errors from becoming too serious, they are of high accuracy. Thus, an important part of the work of the U.S. Coast and Geodetic Survey and the U.S. Geological Survey is establishing a network of lines of levels throughout the domain of the United States such that points whose elevations have been accurately determined will be distant not more than a few miles from any place.

It has been stated previously that in plane surveying the spheroidal shape of the earth is disregarded. An exception to this condition is that of leveling. The requirements as regards accuracy in leveling, particularly that which has to do with the flow of water in pipes or channels, is such that the curvature of the earth must be recognized. The nature and importance of this condition as it affects leveling

work will be discussed in subsequent articles, but as evidence of its significance, it may be stated here that a tangent departs from the mean curved surface of the earth about 0.7 ft in a distance of one mile, whereas, in most leveling work, an error of as much as 0.1 ft in a mile is as great as can be permitted.

Before the present network of accurate levels had been established, it was not possible in many inland places to use sea-level elevations. Usually in such regions, the surface of some adjacent body of water was used as a surface of reference. For example, the City of Chicago used the mean level of Lake Michigan, and St. Louis used the low-water level of the Mississippi River. Such local datums are not necessary now and are being abandoned in favor of the universal sea-level datum.

3-2. General Principles A few general principles and definitions which apply to leveling are illustrated in Fig. 3-1. Here are repre-

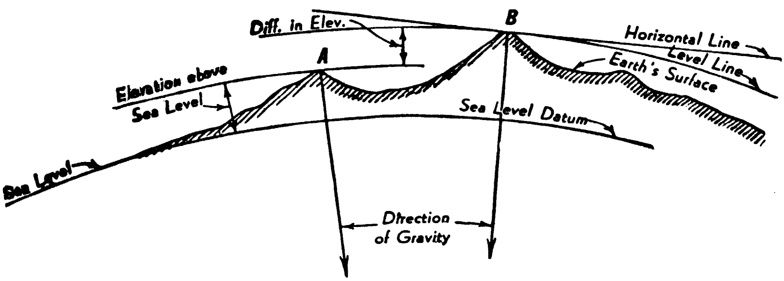


FIG. 3-1. Earth's Curvature and Elevation.

sented mean sea level and the adjacent land surface rising to two summits A and B. Mean sea level is universally regarded as the reference surface to which all elevations are referred. This surface conforms to the spheroidal shape of the earth's surface and is perpendicular to the direction of gravity at every point. It serves as a surface of reference for elevations and is called the *sea level datum.* A cross section of this surface as represented in the figure, is therefore a curved line. The elevation of summit A is measured by the vertical distance along the direction of gravity between the sea-level datum and point A, or between the sea-level datum and a parallel curved surface through A. Likewise, the elevation of summit B is equal to its vertical distance above the sea-level datum. Also, the

difference in elevation between summits *A* and *B* is equal to elevation *B* minus elevation *A*, or it is equal to the vertical distance between the imaginary curved surfaces, parallel to sea level, one passing through *A* and the other through *B*.

From the above considerations we have the following definitions:

A *datum* is a surface of reference, coincident or parallel with mean sea level to which all elevations of a given region are referred.

A level line is a curved line, every element of which is perpendicular to the direction of gravity.

A *horizontal line* is a straight line tangent to a level line at any given point. It is perpendicular to the direction of gravity at the point of tangency only.

Leveling is the process of determining the difference in elevation between two points by measuring the vertical distance between the level surfaces passing through the points.

3-3. Earth's Curvature and Atmospheric Refraction The effects of the earth's curvature and atmospheric refraction are illustrated in Fig. 3-2. For the distance *AD*, a level line parallel with the

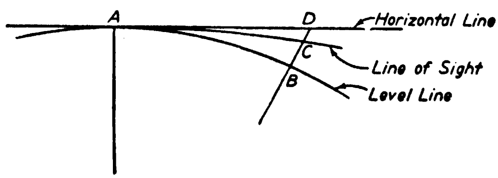


FIG. 3-2. Earth's Curvature and Atmospheric Refraction.

earth's curvature is deflected from a horizontal line the distance *DB*. The amount of this deflection is proportional to the square of the distance *AD;* and, for a distance of one mile, $DB = 0.667$ ft. Evidently this effect is to make a rod reading too great.

The line of sight is deflected from a horizontal line by atmospheric refraction by an amount indicated by the distance *DC*. The amount of this deflection is also proportional to the square of the distance sighted; and, although it varies slightly with atmospheric conditions, it may be taken, with negligible error, to be one-seventh of the magnitude of the earth's curvature and will, as shown, reduce the effect of the earth's curvature by this amount.

The combined effect of earth's curvature and atmospheric refraction can be closely approximated by the expression

$$\text{C \& R} = 0.021S^2 \tag{3-1}$$

where S is the length of the sight in thousands of feet. For sight distances of 100, 200, 300, and 500 ft, C & R becomes 0.0002, 0.0008, 0.0019, and 0.0052 ft, respectively.

3-4. Methods of Leveling Three principal methods are used for determining differences in elevation: *barometric, trigonometric,* and *direct,* or *differential,* leveling.

Barometric leveling makes use of the phenomenon that differences in elevation are proportional to the differences in the atmospheric pressure. Accordingly, the readings of a barometer observed at various points on the earth's surface yield a measure of the relative elevations of these points. The method is explained in Art. 3-36.

The vertical distance between two points in a vertical plane, i.e., difference in elevation, can be determined by the measurement of a vertical angle and a horizontal distance, just as the length of any side in any triangle can be computed from the proper trigonometric relations. This procedure is called *trigonometric* leveling, and a modified form, called *stadia* leveling, is commonly used in mapping. Trigonometric leveling is treated in Art. 3-34.

Vertical distances with respect to a horizontal line (perpendicular to the direction of gravity) may be used to determine the relative difference in elevation between two adjacent points. Since the spirit level, to be described presently, is used to fix the direction of the horizontal line of sight, this method is called *spirit* leveling. It is the most common method of direct or differential leveling and will be the principal subject of this chapter.

3-5. The Engineer's Telescope The engineer's telescope serves the two purposes of fixing the direction of the line of sight and of magnifying the apparent size of objects in the field of view, and its proper use requires a brief description of its essential parts.

The principal features of the engineer's telescope are the objective lens, the cross-wires, and the eyepiece. These parts and their relations to each other are illustrated in Fig. 3-3.

The Objective Lens.—The objective lens forms an image of any object within its field of view. This image is a real image and lies in

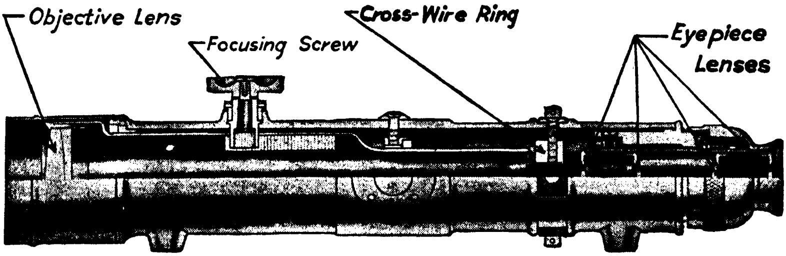


FIG. 3-3. Engineer's Telescope.

a plane, within the telescope, at a distance from the lens which depends on the curvature of the lens surfaces and the distance to the object. The distance from the lens to the image is called the focal distance f, for that particular object. If the object is at a great distance, the image will be formed at a distance from the lens called the *principal focal length F*. These focal lengths will, of course, be different for different telescopes. Thus, the objective lens may be compared with the lens of a camera which forms an image at a definite distance from the lens, i.e., upon the photographic plate or film. Since this distance varies for different objects, a means of changing the focal distance, i.e., of *focusing* the lens, is provided on the telescope by the focusing screw shown in the illustration.

The Eyepiece.—The image which is formed by the objective lens is inverted and small in size. A system of eyepiece lenses is used, therefore, to magnify the image and, in many telescopes, to re-invert or to yield an erect image of the object. The eyepiece then, may be thought of as a microscope with which to view the image formed by the objective lens. An image may be viewed through the eyepiece only when it lies in the focal plane of the eyepiece; and, accordingly, a small amount of focusing is also provided for the eyepiece. An image which has thus been brought into the common focal plane of both the objective and eyepiece lenses will appear to be magnified and distinct.

An eyepiece yields a direct or an inverted image, depending on the arrangement of the lenses. An inverting telescope is superior in its optical properties and is preferred by many engineers. However, the inverted image causes some confusion at first and may easily cause mistakes until the engineer has become thoroughly familiar with its use. Accordingly, an instrument to be used by one person only may have either type of eyepiece, but where two or more instruments are

to be interchangeable, they should all be equipped with the same kind, preferably the erecting type.

The Cross-Wires.—From what has been said, it is evident that, when the image of any object is seen plainly, it lies in the common focal plane of both the objective and the eyepiece lenses; also, that the position of this common plane can be altered by moving (focusing) the eyepiece. Now, if cross-wires are placed in this common focal plane, they will appear to be projected upon the object viewed and will serve to fix the line of sight upon any point of the object. This condition will be effected then, if the cross-wires are fixed in a stationary position and if the eyepiece is then focused upon them, and, finally, if the image of the object to be viewed is brought into the common focal plane by focusing the objective lens.

The proper use of the telescope, then, requires first that the eyepiece be focused upon the cross-wires and then the objective lens is focused upon the object to be observed. Since the position of the cross-wires is fixed, it will be necessary to focus the eyepiece only once for the day's work, or until the position of the eyepiece has somehow been altered.

The manner of mounting the cross-wires is illustrated in Fig. 3-4.

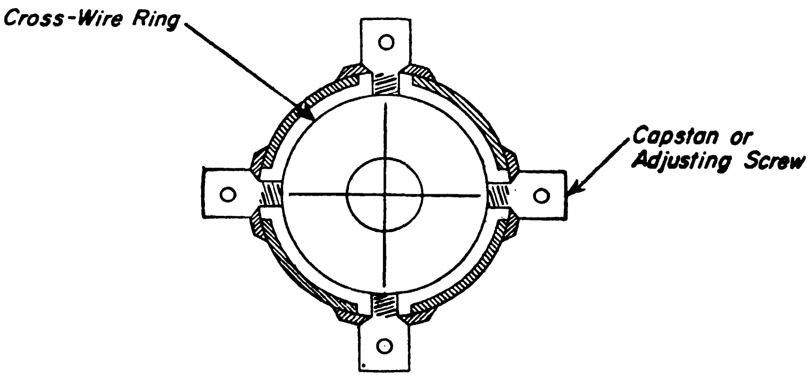


FIG. 3-4. Mounting of Cross-Wire Ring.

The cross-wires consist either of spider web or of finely drawn platinum wire, glued in position upon a heavy brass ring. Also, lines may be etched on a glass diaphragm. Four threaded holes are drilled into the edge of the ring to receive capstan headed screws which are inserted through slots in the barrel of the telescope tube. The heads of

these screws bear against curved washers. By this arrangement the cross-wire ring is held suspended by the capstan screws and within the telescope tube. It is thus held securely and firmly in place, but is subject, as occasion arises, to a small amount of lateral movement by turning the capstan screws. This movement is used for purposes of adjustment, to be described later, and for this reason they are commonly called adjusting screws.

3-6. Parallax From the previous discussion it is evident that when any distant object is viewed through the telescope, the cross-wires will, at the same time, appear plainly only when they lie in the common focal plane of the objective and eyepiece lenses. If the plane of the cross-wires is very close to, but not coincident with, the common focal plane of the lenses, they may appear to be quite distinct, but they will seem to move about on the object with the slightest movement of the eye of the observer. A similar phenomenon is the apparent movement of the window sash with respect to any out-of-door object if the observer moves his head slightly. This condition in the telescope is called *parallax* and is to be prevented by careful focusing of the eyepiece on the cross-wires before viewing any distant objects.

3-7. Magnification The magnification of a telescope is fixed by the ratio of the focal lengths of the objective and the eyepiece lenses. It can be determined closely for any telescope by viewing, at a close range, a graduated rod with one eye looking through the telescope and with the other naked eye. Thus two images are seen, one being the magnified and the other the natural size of the rod. By a suitable adjustment of these images the observer may count the number of divisions on the unmagnified rod which is covered by one of the magnified divisions. This number is the measure of the magnification of the telescope, expressed as diameters. The magnification for telescopes for transits varies from 15 to 30 diameters, and for levels from 25 to 40 diameters.

High magnification, beyond proper limits, is a disadvantage, because it limits the field of view and reduces the illumination or brightness of the objects viewed. Accordingly, the size of the aperture of the objective lens and the qualities of the lens system are as important considerations in a good telescope as is the magnification.

3-8. The Bubble Tube The bubble tube is an important part of most surveying instruments. It consists of a glass tube, the inside surface of which is ground accurately to a curved surface so that a longitudinal section shows, on the upper half, a circular arc of long radius. See Fig. 3-5. The inside of the tube is nearly filled with ether

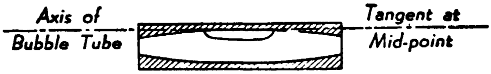


FIG. 3-5. Bubble Tube.

or some nonfreezing liquid, the remaining volume being a vapor space called the bubble. The buoyancy of the liquid lifts the bubble to a position symmetrical with the highest point in the tube, and since this point is on the arc of a vertical circle, the tangent at that point (whether it be at the mid-point of the tube or not) will be truly horizontal and perpendicular to the direction of gravity. Now, if divisions are marked on the tube symmetrical with its mid-point, then when the bubble is centered, the tangent at the mid-point will always be a horizontal line and parallel with the geometrical axis of the tube. Hence, this tangent at the mid-point of the tube is called the *axis of the bubble tube.*

The *sensitivity* of the bubble tube is determined by the radius of the circular arc and is expressed by the seconds of arc subtended by one division (0.1 in.) of the bubble tube. The bubble tubes used on surveying instruments vary, ordinarily, within the range of from 60″ to 10″. A common value for the engineer's level is 15″.

The radius of curvature R, of any given bubble tube, and its corresponding sensitivity v, in seconds of arc, can be found as follows: set up the level carefully and, with the bubble at any given division on the tube, read the position of the cross-wire on a rod at a known distance D. Then, by means of the leveling screws, move the bubble, say, 5 divisions, and read the rod again. The difference between the rod readings, divided by 5, will be an intercept, i, in feet. Then, if the length of one bubble division is represented by d, in inches, the following equation may be stated,

$$R = \frac{D \times d}{i \times 12}. \tag{3-2}$$

Also, the ratio i/D may be taken as the approximate tangent of the arc v, which designates the sensitivity of the bubble tube.

EXAMPLE 3-1.—The intercept found by reading a level rod twice, at a distance of 300 ft, the bubble being moved 5 divisions between readings, is 0.15 ft. The value of a bubble division is 0.1 in. Find the values of R and v.

For these conditions, $i = 0.03$ ft, and $R = \dfrac{300 \times 0.1}{0.03 \times 12} = 83$ ft.

Also, $v = \tan^{-1} = \dfrac{0.03}{300} = 0.0001 = 20''$ (very nearly).

The significance of the sensitivity of the bubble tube is that it expresses the angle through which the line of sight will rotate in a vertical plane if the bubble is moved one division. Conversely, a small vertical movement of either end of the telescope will be accompanied by a relatively large displacement of the bubble in the case of a sensitive bubble tube as contrasted with a relatively small displacement in the case of an insensitive bubble tube. A 10″ bubble tube would be twice as sensitive as a 20″ bubble tube.

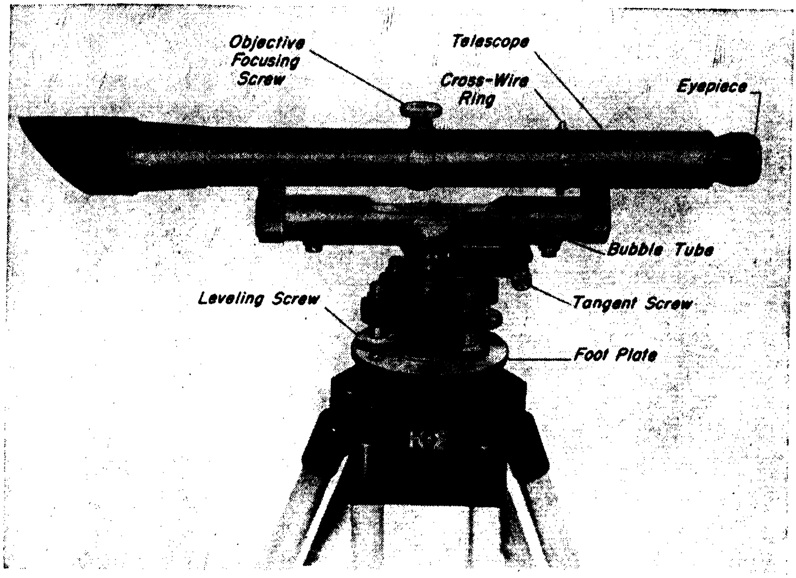
EXAMPLE 3-2. Determine the error in the reading on a level rod if the sight distance is 300 ft, the bubble is one-half division off center, and the sensitivity is 20″.

$$\text{Error} = 300 \times \tan 10'' = 300 \times 0.00005 = 0.015 \text{ ft}$$

3-9. The Dumpy Level The dumpy type of engineer's level is illustrated in Fig. 3-6. Its principal features consist of (1) a telescope which fixes the direction of the line of sight, (2) a bubble tube attached to the telescope, (3) a leveling head which supports the telescope and permits the bubble in the tube to be centered, and (4) a supporting tripod.

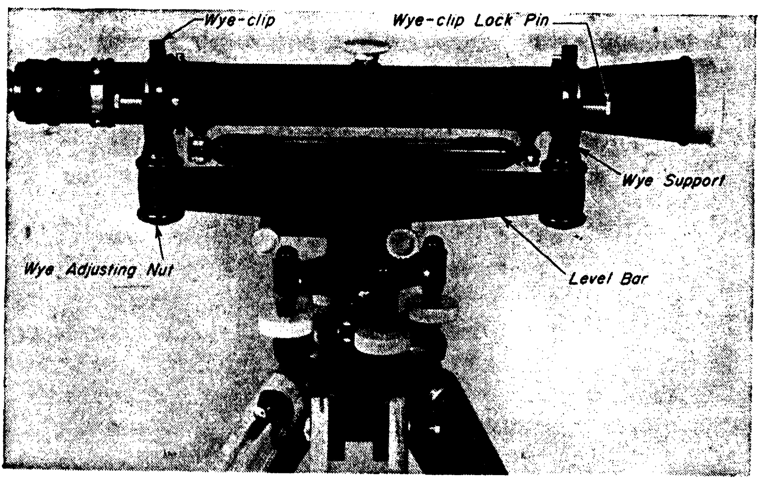
This type is superior in many important respects to the wye type shown in Fig. 3-7. Its principal advantage is that its simplicity of construction results in fewer parts to become worn or displaced with respect to each other. Hence, it does not need adjustment as often and is, accordingly, more dependable in service than the wye type described in the following article. Its cost, both original and for repairs, is less than for the wye level.

The manner of adjusting the dumpy level is sometimes considered a disadvantage. However, the method given in this chapter is believed to be simpler in principle, and actually to require less time, than the wye-level method. Moreover, there are good reasons why a wye level should be adjusted as though it were a dumpy.



Courtesy of Keuffel and Esser Co.

FIG. 3-6. Dumpy Level.



Courtesy of Eugene Dietzgen Co.

FIG. 3-7. Wye Level.

There are no grounds for the belief, sometimes held, that a dumpy level always has an inverting eyepiece.

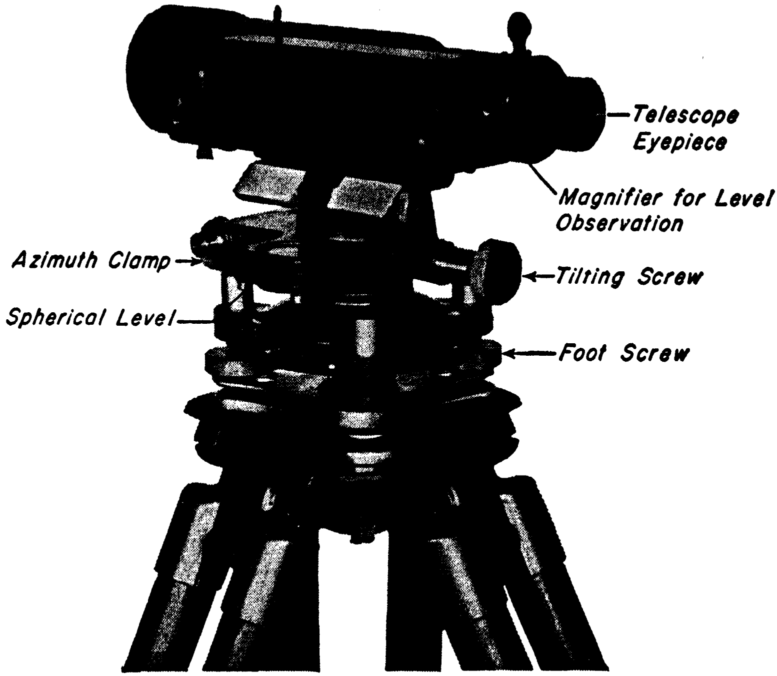
For these reasons, it seems that the dumpy level is superior in every way to the wye level, and there is no reason, other than the influence of custom, why the wye level should be used. There certainly is no justifiable reason why, in the purchase of a new instrument, a wye level should be preferred.

3-10. The Wye Level The wye type of engineer's level received its name from the fact that the telescope rests in Y-shaped supports. The telescope, with the attached bubble, is independent of the other parts of the instrument and may be lifted from its supports and turned end for end; also it may be rotated about its axis in the wyes. See Fig. 3-7.

The remarks of the previous article have shown that the wye level is, in important respects, inferior to the dumpy level. Also, if space were available, it might be shown that the wye level should be used and adjusted exactly as though it were a dumpy. Accordingly, no further attention will be given to this specific type, and the treatment, both of the instruments and methods which follow, may be taken to apply to the wye and the dumpy levels alike.

3-11. Tilting Level A *tilting level* is one whose telescope can be tilted or rotated about a horizontal axis through the use of a hinged joint. This design enables the operator quickly to center the bubble and thus bring the line of sight into the horizontal plane.

Figure 3-8 shows a level instrument of this type. A tilting screw is provided to raise or lower the eyepiece end of the telescope until the bubble as viewed through the aperture on the left side of the telescope is centered. For preliminary leveling of the instrument, a circular level situated on the left side of the telescope is used. In order to minimuze the effects of changes in temperature, the bubble tube is protected by a metal cover. This housing also covers two prisms which are located over the ends of the bubble tube. Hence, the observer sees an image of each end of the bubble. These images are split and separated by a thin line. As the bubble moves in its tube, the images move in opposite directions. The centering of the bubble is effected by causing the two images to become coincident as shown in Fig. 3-9. This device permits the accurate centering of the bubble from the normal observing position of the levelman.



Courtesy of Wild Heerbrugg Instruments, Inc.

FIG. 3-8. Wild N-2 Engineer's Level.

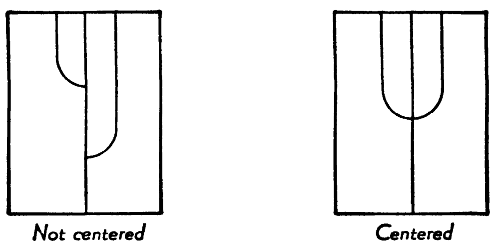


FIG. 3-9. Ends of Bubble as Seen in the Reflector.

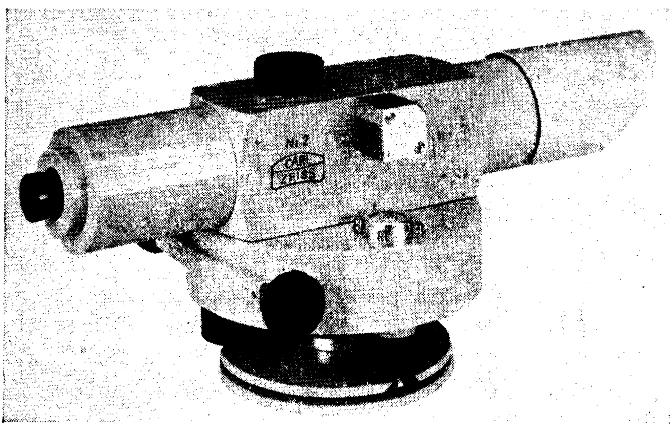
There is no need to walk about the level in order to view the bubble.

A distinguishing feature of the Wild N-2 level is the fact that both the inside upper and lower surfaces of the bubble tube are ground to the arc of a circle of the same radius and that the telescope can be rotated 180° about its optical axis together with the bubble tube. This permits checking the relationship between the

line of sight and the axis of the bubble tube from a single setup of the level. The subject of adjustment is treated in Art. 3-22.

This instrument is supported by three foot screws which are manipulated to effect a preliminary leveling with the use of the spherical level. The magnification is 24 or 28 diameters. Levels of this make are transported in handy, metal cases which protect the instrument from dust, humidity, and damage.

3-12. Automatic Level The *automatic level*, shown in Fig. 3-10, makes use of a pendulum to maintain continuously and automatic-



Courtesy of Keuffel & Esser Co.

FIG. 3-10. Automatic Level.

ally the line of sight in a horizontal position. This level has made possible pronounced economies in leveling operations because of the speed with which it can be used. The instrument has no tubular spirit level and no tilting screw. The observer merely centers a circular bubble. Then the pendulum functions to define a horizontal line of sight, and the rod readings are obtained. In addition to accelerating leveling operations, the automatic level is very useful under conditions of unstable ground and wind when the bubble of a spirit level must be continuously under observation to make certain it is centered.

3-13. The Hand Level For many purposes a hand level gives results with sufficient accuracy, and it is far more convenient and

rapid in use than the engineer's level. One kind, illustrated in Fig. 3-11, enables the observer to determine a horizontal line of sight quickly by noting the apparent position of a bubble with respect to a dividing line fixed in the field of view.

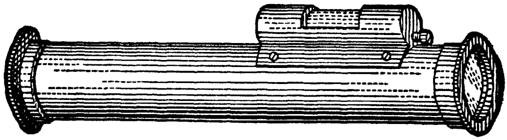


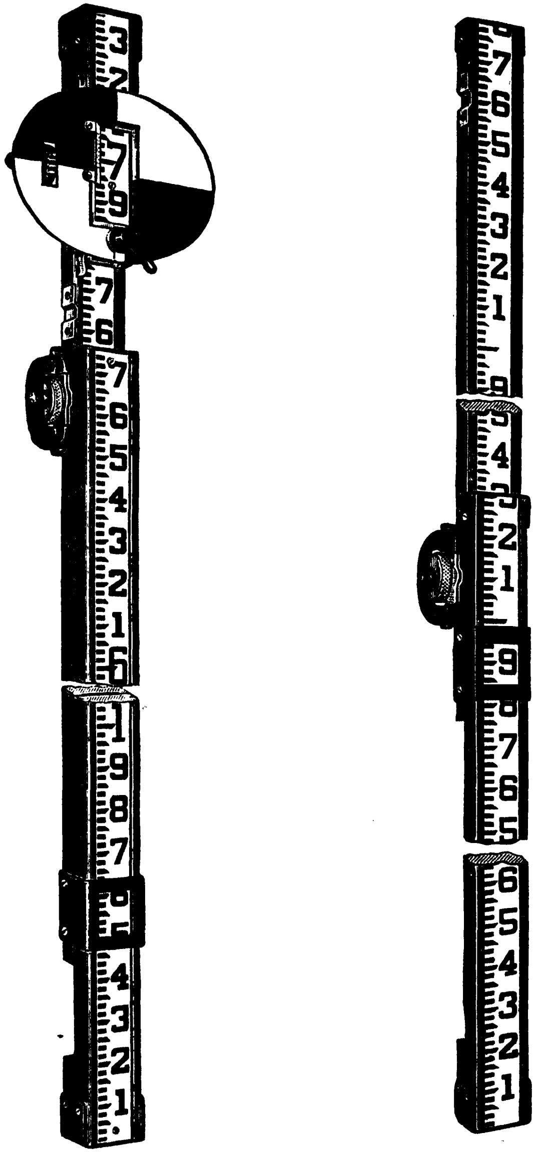
FIG. 3-11. Hand Level.

If the lengths of sight are limited to a maximum of 50 ft, a line of levels may be carried a distance of 400 or 500 ft with an estimated error of about 1 ft; and if the lengths of sight are decreased below or increased above 50 ft, the resulting accuracy will be greater or less, respectively, than that mentioned.

3-14. Level Rods Level rods are constructed in a variety of forms and patterns, but that which has gained by far the widest use is the so-called Philadelphia rod, illustrated in two forms in Fig. 3-12. It is graduated in units of feet, tenths, and hundreths, the divisions being indicated by alternate black and white spaces of 0.01-ft dimension. A little study of the markings makes evident the exact subdivisions intended, and the pattern is excellently devised to permit readings to be taken quickly, easily, and with small chance for mistakes.

Form (a) is supplied with a target on which may be noted an auxiliary scale called a *vernier* by which subdivisions to thousandths of a foot may be read. (This device is described in Art. 3-15.) The pattern, however, is sufficiently legible to be read directly from the instrument. Furthermore, since rod readings are not taken to thousandths except in precise work, the target is seldom used, except for sights of unusual length or when vegetation or other obstructions to vision make the reading difficult. Accordingly, many levelmen remove the target, in which case all readings are taken from the instrument, and the rod, capable of such use, is called a *self-reading* rod. Obviously both forms (a) and (b) are of this type.

Form (a) is in two sections which, when folded together, form a continuously subdivided length of 7 ft. For readings requiring a greater length, the rod is extended to its full length of 13 ft and



(a) (b)

Courtesy of Keuffel & Esser Co.

FIG. 3-12. Level Rods.

clamped in this position. Form (*b*), in three sections, may be folded to a length of about 5 ft, making it a more convenient rod to handle and to transport. It may be extended by successive lengths to its full length of 12 ft.

3-15. The Vernier The vernier is a device for reading subdivisions of a graduated scale. Such a scale may be rectilinear, as that of a level rod, or it may be circular, as that of a transit. The principle is the same in either case.

The Linear Scale.—Fig. 3-13a shows a decimal scale with a vernier alongside that is movable with respect to the scale. The vernier is divided into 10 parts, which cover 9 parts on the scale. Hence, the value of one vernier division is equal to 9/10 of one scale division; therefore, in reading upward from 6.00 on the scale, the first division on the vernier falls 1/10 of a division short of the first mark on the scale. Likewise, the second division mark on the vernier falls 2/10 of a division short of the second scale mark, etc.

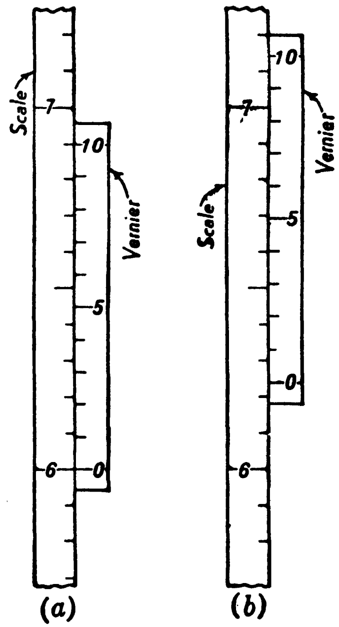


FIG. 3-13. Linear Scale with Vernier.

Now it will be seen that, if the vernier is moved upward until the first mark on the vernier coincides with a scale mark, the zero, or index, of the vernier will have moved upward 1/10 of a scale division; in which position the vernier reading would be 6.01. Similarly, if the vernier is moved upward until the second mark on the vernier coincides with a scale, then the index will have moved 2/10 of a division and the reading would be 6.02; etc.

In Fig. 3-13b it is seen that the index of the vernier has been moved upward (above 6.00) past two scale divisions and that the fourth mark of the vernier coincides with a scale division. Accordingly, the reading of the vernier is 6.24.

Evidently, the smallest subdivision which can be read with this

vernier is 1/10 of a scale division and is determined by the number of parts on the vernier. For example, if a vernier were arranged with 15 parts to cover 14 parts on the scale, then the smallest subdivision which could be read with such a vernier would be 1/15 of a scale division. Hence, the general principle of all verniers may be stated that, if n equals the number of vernier divisions, and if these n *vernier* divisions cover $(n - 1)$ scale divisions, the smallest subdivision of the scale that can be read with the vernier is equal to the value of a scale division divided by n. Or,

$$D_s = \frac{s}{n}. \tag{3-3}$$

in which D_s is the smallest subdivision of the scale that can be read, s is the value of a scale division, and n is the number of divisions on the vernier.

3-16. The Rod Level The rod level is a device for plumbing the rod. One form is shown in Fig. 3-14.

Courtesy of Keuffel & Esser Co.

FIG. 3-14. Rod Level.

3-17. Care of Equipment The student is responsible at all times for the proper care of the instruments assigned to him. Surveying instruments have many parts of precise and delicate workmanship which, if once injured, are seriously affected in their operation, and repairs are highly expensive. They should be handled and manipulated carefully under all conditions. Following are a few suggestions:

Taking Instruments to the Field.—The instrument box is made to fit the instrument neatly so as properly to support and protect it; hence, in removing it for the first time, the student should observe how it is placed so it may be returned properly to the box. When unscrewed from its base in the box, or from the tripod, the instrument should be supported by the hand or arm held under the leveling head. Be careful that the head of the instrument is securely attached to the tripod. The instrument on its tripod should never be set up on the floor of a building unless the shoes of the tripod are securely held. The authors' experience with students has been that, when an instrument is permitted to fall to the floor or to the ground, the cost of repairs will be from $75.00 to $150.00 for a level and considerably more for a transit.

The instrument should not be carried on the shoulder within a building, nor through hallways and doors.

Screws and Clamps.—The threads of the screws and clamps are easily injured by unnecessary pressure used in tightening them. In all cases, when manipulated properly and when in good condition, they will turn easily and should be brought to a snug bearing only. Any further tightening serves no purpose except to injure them. Extreme changes in temperature are likely to cause undue stresses in the different parts of the instrument because of their unequal rates of expansion or contraction. Hence, when instruments are brought from a cold, outdoor temperature into a warm room, or vice versa, the leveling screws and clamps should be loose enough that no injury will result.

If the bearings, screws, or clamps do not turn easily, they should be cleaned with a good solvent such as alcohol or gasoline. Any interior bearings or threads should then be oiled lightly with watch oil, any excess being wiped away. An exposed oiled surface will collect dust, so exposed threads should not be oiled. If any bearings or screws do not turn smoothly after cleaning and oiling, probably the parts have been injured and should be returned to the maker for repairs.

When in its box, the head of the instrument should be evenly leveled all around and the clamps set snugly to prevent the displacement of the instrument in transport. When being carried on its tripod, the clamps of the instrument should be loose so that, if it should strike any obstruction, the different parts can yield and thus possibly avoid injury.

The Lenses.—The quality of the telescope lenses may be seriously and permanently injured by unnecessary or improper abrasion in cleaning. Only when cleaning is plainly necessary should it be attempted and then only a soft material should be used. Ordinarily, moisture on the lens should not be wiped off but permitted to evaporate.

Moisture.—The instrument should be protected as much as possible from moisture and dust. A canvas hood is provided for this purpose when the instrument is in field use and should be carried to the field in threatening weather. When the instrument has been exposed to moisture it should be wiped dry (except the lenses) before it is replaced in its box or locker.

Exposure to Danger.—An instrument should never be left unprotected on a sidewalk or railway, in a street, or pasture, or any place where persons, vehicles, or animals might injure it.

3-18. Theory of Differential Leveling Consider a level set up on a horizontal surface, as illustrated in Fig. 3-15, with a rod held

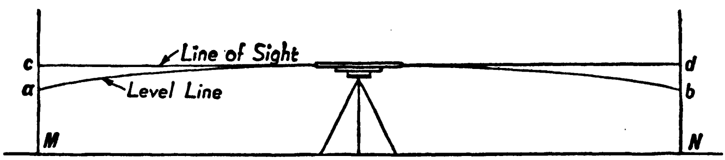


FIG. 3-15. Theory of Leveling.

at an equal distance on either side of the instrument at points M and N, respectively. Evidently the difference in elevation between these points is zero. Consider also that the level line passing through the telescope cuts the rod at points a and b, and that the line of sight cuts the rod at points c and d. It is evident that $a - b = c - d = 0$. Hence, we see that, for any two points which are equidistant from the instrument, the line-of-sight readings give the true difference in elevation regardless of the curvature of the earth or atmospheric

refraction. The deflection of the level line from the line of sight is quite small for the ordinary lengths of sight used in leveling, being about 0.002 ft in 300 ft. For these two reasons the effect of the curvature of the earth and refraction will be very small for the usual conditions of ordinary leveling.

When the curvature of the earth and refraction are neglected, the theory of leveling is simple. With the level set up at any place, the difference in elevation between any two points within proper lengths of sight is given by the difference between rod readings taken on these points. By a succession of instrument stations and related readings, the difference in elevation between widely separated points is thus obtained.

By referring to Fig. 3-16, the common terms used in leveling may be defined.

A *benchmark* is a permanent object of known elevation. It should be definite and located where it will have the smallest likelihood of being disturbed. Examples are a metal or concrete post planted in the ground, a notch cut in the root of a tree, a spike driven in a tree or pole, a definite corner of the masonry of a bridge or building, and a water hydrant.

A *turning point* is a temporary benchmark, i.e., a definite, firm object whose elevation is determined in the process of leveling, but which has served its purpose as soon as the necessary readings have been taken upon it. Frequently, the head of the rodman's hand-axe, which has been driven in the ground, is used as a turning point.

A backsight is a rod reading taken on a benchmark or turning point of known elevation. It is the vertical distance from the benchmark or turning point to the line of sight.

The *height of instrument* is the elevation of the line of sight. It is found by adding the backsight to the elevation of the point on which the reading is taken.

A *foresight* is a rod reading taken on a turning point or other object whose elevation is to be determined. It is the vertical distance from the line of sight to the point observed.

3-19. Field Procedure In setting up the level, the tripod legs are adjusted so that the leveling head base is approximately level, and then the legs are firmly pushed into the ground. On a steep side hill, two legs are placed on the downhill side and one uphill. The foot screws (of dumpy and wye levels) are loosened and the leveling

head is then shifted until one pair of foot screws is parallel with the general direction of the line of levels. The foot screws are operated in diagonal pairs, and in each pair one screw is turned inward and the other outward, the proper direction being determined by the fact that the bubble moves in the same direction as the left thumb. If one screw is turned more rapidly than the other, either the pair will bind, or the leveling head will become unstable and may wobble from side to side. Either condition is remedied by turning one screw only until the proper bearing of both has been re-established.

The bubble is centered first over one pair of screws and then over the other pair, after which the telescope is sighted on the rod and the bubble is given its final centering. The beginner is likely to waste much time in the first two steps, for which the bubble need not be centered more nearly than within two or three bubble divisions. It is only when the rod reading is taken that the centering of the bubble becomes a source of error. The beginner is also likely to spend much time carefully centering the bubble only to find, when he sights the rod, that his line of sight is either below the bottom or above the top. The time thus spent is wasted and could largely have been prevented by quickly assuring himself, by roughly sighting the rod when the instrument is first set up, that it is not too high or too low.

The field procedure of running a line of levels is illustrated in Fig. 3-16, and the corresponding field notes are shown in Fig. 3-17.

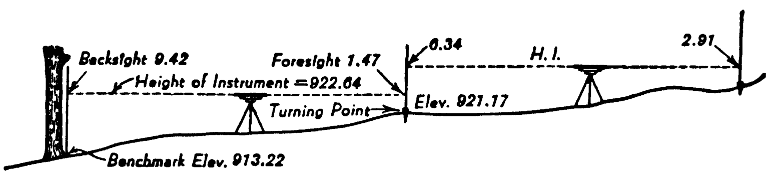


FIG. 3-16. Differential Leveling.

The levelman sets up the instrument at a proper distance from the benchmark. He prepares his notebook with the headings, as shown, and enters the usual data regarding the party organization, date, weather, etc., also the elevation and careful description of the benchmark from which the line of levels begins. The rodman holds his rod on the initial benchmark, whose elevation is 913.22 ft above sea level, and the levelman finds the backsight reading to be 9.42 ft. This reading is recorded in the backsight (B.S.) column

and on the line which pertains to the object observed. The rodman proceeds to a point properly located with respect to the instrument and the line of levels, selects or fixes a good turning point and holds his rod on it. While the rodman is going forward, the levelman adds the backsight reading to the elevation of the benchmark to determine the height of instrument (H.I.), which is 922.64.

DIFFERENTIAL LEVELING FROM TURKEY RUN CREEK TO ROUTE 24 — Locker 47 — J. E. Slattery ⊼, W.D. Fooks Rod, June 1, 19 — 33

Sta.	B.S.	H.I.	F.S.	Elev.	
B.M.$_1$	9.42	922.64		913.22	Spike on Root of 10 in. Oak
T.P.	6.34	927.51	1.47	921.17	200 ft east of Highway Bridge over Turkey Run Creek
T.P.	10.15	934.75	2.91	924.60	
T.P.	9.78	943.39	1.14	933.61	
	5.38	948.04	0.73	942.66	
B.M.$_2$	—		3.07	944.97	
	41.07		9.32	913.22	
	9.32	Check		31.75	Boulder at S.E. Fence Corner at Cross-roads, on Route 24
	31.75				

FIG. 3-17. Field Notes for Differential Leveling.

The levelman then observes the rod and finds this foresight reading to be 1.47 ft. This is recorded in the foresight (F.S.) column and on the line pertaining to this turning point (T.P.). This reading subtracted from the H.I. gives the elevation of the T.P., 921.17. Thus the elevation of the first T.P. has been determined with respect to B.M.$_1$

The levelman now goes forward, selects a proper position and sets up his level. The rodman remains at the turning point until the levelman is ready, when he again holds his rod to be read by the levelman. This reading, 6.34, is a backsight and is entered in that column and on the line which pertains to the first T.P. It is added to the elevation of the T.P. to determine the new H.I., 927.51. The rodman now proceeds to establish another turning point. The levelman finds the new foresight to be 2.91, and so the work proceeds until a second benchmark, B.M.$_2$ is reached. Here the foresight is 3.07 and the elevation is 944.97.

Thus the elevation of B.M.$_2$ has been determined with respect to B.M.$_1$ and obviously the process could, if necessary, be continued to obtain the elevations of other more widely separated points.

3-20. Precautions The subjects of errors and mistakes are treated below, but three remarks should be made here.

1. *Centering the Bubble.* It has been said that the condition of the bubble being centered is a source of error only when the rod is sighted, and time need not be wasted centering it at other times. At that instant, however, it is a principal source of error, and great care should be taken that the bubble is centered when the rod is read. The experienced levelman habitually looks at the bubble both before and after reading the rod to assure himself that the bubble was centered at that moment.

2. *Keeping the Rod Plumb.* The rodman must be careful to keep the rod plumb whenever a reading is taken.

3. *Equal Sights.* The backsight and foresight distances should be equalized insofar as field conditions permit. This is largely the responsibility of the rodman and is accomplished by estimation or by pacing. Sight distances should be limited to 300 ft.

3-21. Field Notes The main body of the field notes in leveling consists in a tabulation of the readings and of the computed elevations, for which fairly definite forms have been generally adopted, and unless the student masters these before going to the field, he will be more confused in keeping the notes than in attending to his other duties. The standard forms for the usual kinds of leveling work are given in the following pages and, although these should be carefully studied, the student should remember what has been said about field notes generally in Art. 1-8. A few remarks which pertain to leveling in particular are given in the following paragraphs.

Checking the Level Notes.—The possibilities for mistakes in the additions and subtractions in the notes make it necessary to apply a check to these computations before any reliance can be placed on the final values. It may be noted that backsights are always added and foresights are always subtracted, and on this account the columns are commonly called (+) and (−), respectively. Obviously, the difference in elevation between the initial and the final benchmarks is equal to the difference in the sums of the backsight and

foresight readings. Accordingly, no level notes may be regarded as complete until this check has been applied. (See Fig. 3-17.)

It is especially important that each benchmark be so clearly and completely described that anyone not familiar with the vicinity can find and use it at any subsequent time, possibly years later.

Checks on level work are frequently made in the field by closing circuits; consequently, the elevations of all benchmarks should be computed in the field as the work proceeds.

All records should be made in their proper place in the regular field notebook. The student is sometimes inclined to make temporary entries in the back of the book or on a loose-leaf sheet of paper, expecting to rectify and improve the record when it is copied into the notebook. He should avoid such bad practice, knowing that in actual work it would not be permitted.

Many times the elevation of a benchmark on one page in the notebook is copied from, or is related to, the same data on another page. In such a case, a cross-reference is necessary to facilitate comparisons at any time.

3-22. Adjustment of the Engineer's Level In a previous article it has been stated that an important advantage of the dumpy as compared with the wye level is the fact that its adjustments are more durable and dependable. To the initiated, it may be said, further, that if the points of support of a wye level become unevenly worn, no amount of adjustment by the usual wye level methods will reveal that condition; accordingly, the experienced engineer occasionally tests his wye level by the dumpy or "two-peg" test to assure himself that the instrument is in correct adjustment. Furthermore, the many parts of the wye level will be subject to less displacement if the wye clips and the telescope are always left in place.

For these good reasons the student and practicing engineer will secure better results if he will adjust his level, whether it be a dumpy or a wye, by the simple method given below.

From what has been said previously about the care of instruments, it should be recalled that, in making the adjustments, undue pressure should not be exerted on the capstan screws, because their threads may be injured. Also, all adjusting screws should be brought to a firm, but not tight, bearing before the work is completed.

Many engineers do not make the tests or adjustments of their instruments often enough to retain a familiarity with the methods.

Because the job is somewhat delicate, they are inclined to neglect it until it is desirable to send the instrument back to the factory. Such practice is to be deplored, and it is believed that if the beginner once thoroughly understands the simple relations and necessary operations, he should feel no hesitation at any later time to make the tests and adjustments as frequently as may be desirable. Accordingly, the student should regard the topic of the adjustments as one of the highest importance.

The student will be aided to a better understanding of the adjustments of the level if he will, so to speak, overlook the mechanical parts of the instrument and think first only of the simple essential relations between the elementary lines that should exist. Thus, he should understand that these line relations are not affected by any particular position in which the instrument, as a whole, happens to be. They remain fixed with respect to each other, whether the instrument is being carried on the shoulder or being carefully set up and leveled.

Definitions.—These lines may be defined and their proper relations to each other stated as follows:

1. *The line of sight* is the line fixed by the intersection of the cross-wires and the center of the objective lens.

2. *The axis of the bubble tube* (Art. 3-8) is the tangent to the bubble tube at its mid-point.

3. *The vertical axis* is the axis of the spindle about which the telescope and bubble tube rotate.

Relations.—The relations which should exist between these lines are as follows:

1. Insofar as the accuracy of results is concerned, there is but one essential relation, namely, that *the line of sight shall be parallel with the axis of the bubble tube;* but as a matter of convenience two other relations are desirable:

2. *The horizontal cross-wire should lie in a plane perpendicular to the vertical axis;* and

3. *The axis of the bubble tube should be perpendicular to the vertical axis.*

3-23. Adjustment of the Dumpy Level

1. ADJUSTMENT OF THE CROSS-WIRE RING

Relation.—To make the horizontal cross-wire lie in a plane perpendicular to the vertical axis.

Test.—Set up the level and sight some definite fixed point in the field of view. The bubble does not need to be exactly centered for this test and so, if necessary, the telescope may be moved up or down with the foot screws, and to the right or left on its vertical axis, until one end of the horizontal cross-wire is fixed on the point (see Fig. 3-18).

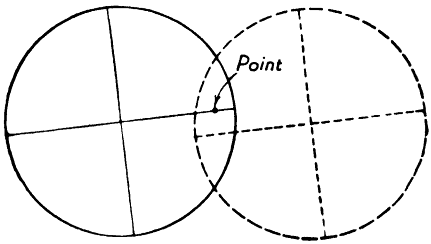


FIG. 3-18. Adjustment of Cross-Wire Ring.

Then turn the telescope about the vertical axis until the other end of the cross-wire has reached the second position shown by dashed lines in the illustration. If the relation being tested exists, the point will appear to move along and will remain on the horizontal cross-wire. If not, as shown in the figure, the point will appear to move off the cross-wire.

Adjustment.—The adjustment is made by slightly loosening both pairs of capstan screws which hold the cross-wire ring in position, and by turning the ring with pressure of the fingers or by tapping lightly with a pencil. When the adjustment is complete, the point will remain on the cross-wire as the telescope is moved slowly from side to side.

2. Adjustment of the Bubble Tube

Relation.—To make the axis of the bubble tube perpendicular to the vertical axis.

Test.—Center the bubble carefully over both pairs of leveling screws and bring it exactly to center over one pair. Then turn the telescope about, end for end, over the same pair of screws. If the correct relation exists, the bubble will remain centered. If not, it will move away from the center and the amount of the movement indicates double the error of adjustment. This relation between the axis of the bubble tube and the vertical axis both before and after re-

versal is shown in Fig. 3-19, (*a*) and (*b*), from which it is evident that the assumed error in the relationship is represented by the angle β, and that after reversal the inclination of the axis of the bubble tube is equal to 2β.

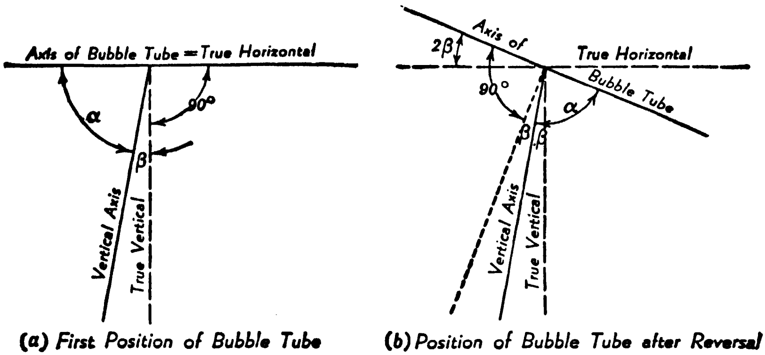


FIG. 3-19. Effect of Reversal of Bubble Tube.

Adjustment.—The adjustment is effected by bringing the bubble back halfway with the adjusting screw at one end of the bubble tube. The bubble is then brought to center with the foot screws. In other words, the bubble is adjusted by the capstan screw, the amount represented by the angle β, and then the vertical axis is moved through the same angle into its vertical position by means of the foot screws. Accordingly, if the above adjustment has been done correctly, the bubble will now remain centered both before and after reversal, thus proving that its axis is perpendicular to the vertical axis.

The test should be repeated to verify the condition.

3. Adjustment of Line of Sight

Relation.—To make the line of sight parallel with the axis of the bubble tube.

Test.—Set the level up and drive one stake, *A*, at a distance of about 150 ft, and another stake, *B*, in the opposite direction at the same distance (see Fig. 3-20). Take a rod reading *a* on the first stake and a reading *b* on the other stake. Obviously the difference in the readings, $b - a$, is the true difference in elevation between the two stakes, regardless of any error in the instrument.

Now set the instrument up near stake *A* so that the eyepiece will swing very close, i.e., within $\frac{1}{2}$ in. of the rod, and read the rod by

observing through the objective lens. The levelman will not be able to see the cross-wires, but a pencil held by the rodman will indicate the reading at c. This reading will be without error. Hence, if the true difference in elevation $(b - a)$ is added to the reading c, the

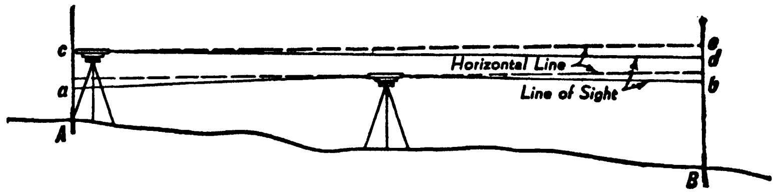


FIG. 3-20. Adjustment of Line of Sight.

sum will be the reading e, which may be called the *computed correct reading* on stake B that would be obtained if the line of sight were truly horizontal and therefore parallel with the axis of the bubble tube, i.e., in perfect adjustment. If the reading on b is less than on a, then the stake at B is higher than A, and, hence, the true difference in elevation $(a - b)$ will be subtracted from c to find the *computed correct reading* on stake B. Now when the levelman reads the rod on stake B, he has the direct evidence as to both the amount and the direction of the inclination of the line of sight for the distance between the two stakes.

Thus, assume readings as follows: $a = 2.04$, $b = 5.16$, and $c = 4.73$. Then $b - a = 3.12$ ft, the true difference in elevation between the stakes A and B. Since b is larger than a, the stake B is lower than A, and therefore $c + (b - a) = e$, or $4.73 + 3.12 = 7.85$ ft, the computed correct reading on stake B. Then the reading $d = 7.81$ is taken, and it is evident that the line of sight is inclined downward $7.85 - 7.81 = 0.04$ ft for the distance between the two stakes.

Adjustment.—The adjustment is made with the instrument still in position at A, by keeping the bubble centered and by moving the horizontal cross-wire until it apparently cuts the rod at the correct reading. In the case assumed, the cross-wire would be moved until the rod reading is 7.85 ft.

This is accomplished by movements of the vertical pair of capstan screws which hold the cross-wire ring. One screw is loosened slightly and the opposite one is tightened a like amount, thus moving, apparently, the position of the horizontal cross-wire on the level rod. This is continued until the desired reading has been obtained. The

horizontal pair of screws is left untouched to avoid disturbing the first adjustment.

3-24. Adjustments of the Wye Level

1. Adjustment of the Cross-Wire Ring

Relation.—The horizontal cross-wire should lie in a plane perpendicular to the vertical axis.

Test and Adjustment.—Same as for the dumpy level. Art. 3-23.

2. Adjustment of the Bubble Tube

Relation.—The axis of the bubble tube should be parallel with the wye supports.

Test.—Loosen the clips over the telescope and level the bubble. Carefully lift the telescope from the wyes, turn it end for end, and replace it in the wyes. If the bubble remains centered, no adjustment is necessary.

Adjustment.—One half the apparent movement of the bubble is adjusted by the capstan screws at one end of the bubble tube. Relevel with the foot screws and repeat the test.

3. Adjustment of the Line of Sight

Relation.—The line of sight should coincide with the axis of the wyes.

Test and Adjustment.—Sight on a definite point and rotate the telescope one-half turn in its wyes. If the horizontal cross-wire remains on the point after rotation, no adjustment is necessary. If the horizontal wire moves up or down from the point, it is adjusted one half the apparent error by means of the vertical pair of cross-wire adjusting screws. After adjustment, repeat the test for verification. If the vertical cross-wire moves to the right or left of the point, it may, if desired, be adjusted one half the apparent error, by the horizontal pair of adjusting screws. Replace the clips over the telescope.

4. Adjustment of the Wyes

Relation.—The axis of the wyes should be perpendicular to the vertical axis.

Test.—Same as for the bubble tube of the dumpy level. Art. 3-23.

Adjustment.—Correct one-half the apparent error by means of the

capstan screws at one end of the level bar. Relevel with the leveling screws and repeat the test.

3-25. Sources of Error in Differential Leveling The principal sources of error which affect the results of leveling are listed below. Each is discussed briefly as to the nature of the source, the magnitude of the error, and the means by which it may be minimized or eliminated.

One condition of leveling should be noted. In the usual procedure there will always be an equal number of backsights and foresights insofar as turning points and benchmarks are concerned. Since a backsight is always added and a foresight is always subtracted, any systematic error present in both readings thus becomes accidental, since the error is as likely to be positive as negative in its effect. In fact, all of the systematic errors of leveling are thus rendered accidental or are to some extent modified in their effects.

1. *Nonadjustment of the Instrument.* The purpose of the first adjustment is to cause the horizontal cross-wire to lie in a horizontal plane so that a rod reading taken anywhere along its length will be the same. If readings were always taken at the intersection of the cross-wires, this adjustment would be immaterial.

The second adjustment is merely for the convenience of the levelman and in no way affects the precision of results, for when the telescope is turned the bubble is always recentered.

The third adjustment is the one important adjustment which affects the precision of results, because, when the bubble is centered, if the line of sight is not parallel to the axis of the bubble tube, it will not give a correct reading of the rod. This error is systematic, but becomes accidental in the process of leveling.

The magnitude of the error depends on the inclination of the line of sight with respect to the axis of the bubble tube and is eliminated completely to the extent that backsight distances are made equal to foresight distances. Accordingly, this source of error is minimized by keeping the instrument in good adjustment and by equalizing backsight and foresight distances. This latter end may be attained within proper limits for most work by estimation. For more careful work the distances may be paced or obtained by stadia readings.

2. *The Bubble Not Centered.* If the bubble is not centered when the rod is read, an accidental error in the reading results. The magnitude of the error depends on the sensitiveness of the bubble tube.

Thus, for the usual bubble tube of 15-sec sensitiveness, the line of sight is displaced 0.02 ft at a distance of 300 ft when the bubble is one division off center. This source of error is minimized by carefully watching the bubble at the time the rod reading is taken. The experienced levelman looks at the bubble both before and after reading the rod to make sure the bubble was centered when the reading was taken.

3. *Incorrect Reading of the Rod.* This source of error is due to the fact that the eye is not able to judge exactly where the horizontal cross-wire apparently cuts the rod or the target. It is an accidental error and its magnitude depends on the distance to the rod, the qualities of the telescope, parallax, the weather conditions, etc. The error should not exceed 0.005 ft at a distance of 300 ft, and this may be taken as about the maximum distance for good results of ordinary precision. If weather conditions are adverse, the length of sight is shortened correspondingly. When long sights are necessary, as when crossing a wide river, the mean of a series of readings is taken.

4. *The Rod Not Plumb.* If the rod is not plumb when a reading is taken, a positive systematic error results. The magnitude depends on the size of the rod reading, i.e., it is greater near the top than near the bottom of the rod. Its value may be estimated by use of the approximate formula for reducing slope measurements in taping (Art. 2-9). The error is systematic as long as, in a series of readings, the backsight readings are larger than foresight, or vice versa. These conditions result in leveling up (or down) a hill. Thus in running a line of levels up a hill, backsight readings are likely to be increased more than foresights from this source, and the elevation of a benchmark on top will be too great. Also, in running a line of levels down a hill, the elevation of a benchmark at the bottom will be too small. Therefore, in a circuit of levels in which the initial and final benchmarks have about the same elevation, this source of error is rendered accidental and hence compensative. However, any benchmarks along the line which are much above or below the initial point will be affected as stated above.

This source of error is minimized by carefully plumbing the rod for all readings. This is done by the rodman standing squarely behind the rod and balancing it between the finger tips of both hands. When the wind is blowing, this is more difficult to do. For this condition, and in fact at all times when readings are taken near the top

of the rod, it is a good practice for the rodman to "wave the rod" slowly toward and away from the level, and the levelman takes the minimum reading. The rod is plumbed in the direction transverse to the line of sight by means of the vertical cross-wire.

For more careful work a rod level is used to plumb the rod. See Fig. 3-14.

5. *Parallax.* Parallax is a source of accidental error. See Art. 3-6. It is eliminated by focusing the eyepiece on the cross-wires.

6. *Curvature of the Earth and Atmospheric Refraction.* The nature of the effect of the earth's curvature and atmospheric refraction has been discussed in Art. 3-3. At 300 ft this source of error is 0.002 ft, which is quite negligible so far as a single reading is affected. In the long run, however, if a considerable number of unbalanced sights are taken, i.e., foresights longer than backsights, or vice versa, an appreciable error may result.

This error is eliminated to the extent that foresight and backsight distances are made equal.

If a wide river or ravine necessitates a long sight, the error may be eliminated by a calculated correction as indicated above, or by the method of reciprocal leveling (see Art. 3-31).

7. *Incorrect Length of Rod.* Obviously a rod of incorrect length will cause a systematic error, but this again is rendered accidental in the process of leveling, provided the differences in elevation are not great. However, if a line of levels is carried from the bottom to the top of a hill (or vice versa), a serious error will result if the rod has even a small error in its length. This error is similar to an error in the length of a tape in taping.

The error is to be minimized by testing the length of the rod from time to time by comparing its length with a steel tape. If necessary, computed corrections can then be made to benchmark elevations.

Sometimes an error is caused by improper joining of the sections of the rod. If this condition is detected, the rod should be discarded. This condition will also result if, in extending the rod, the rodman is not careful to extend the rod its full length.

8. *Settling of the Instrument.* In swampy ground, mud, or melting snow the level may settle in the interval of time between a backsight and a foresight reading. If so, a systematic error results which, in the ordinary process of leveling, is not rendered accidental, because the foresight reading will always be too small. For this condition extra precautions are taken to insure the stability of the level, or

the error can be rendered compensative if, on alternate setups, the foresight is taken before the backsight.

9. *Poor Turning Points.* Either because of soft ground or otherwise, a turning point may be faulty such that the bottom of the rod will not have the same elevation for the different readings taken upon it. For example, a turning point in the middle of a sidewalk would be a poor turning point unless it was definitely marked, because the rodman might be called away from the spot and upon returning he would not be able to recover the exact point. Also, even for levels of ordinary precision, a turning point is never taken on the bare ground because of the indefiniteness and instability of such a point. Accordingly, this source of error is minimized by taking turning points on such definite and stable objects as the top of a stake, a solid stone, or the head of a steel pin driven in the ground.

10. *Heat Waves.* Under certain conditions there are noticeable "heat waves" which affect the precision of the rod readings. This is an accidental error to be minimized by limiting the lengths of sights. Under extreme conditions, leveling work is abandoned until the heat waves have subsided.

11. *Wind.* A high wind shakes the instrument, making it difficult to keep the bubble centered and to read the rod correctly. This is also an accidental error to be minimized by shortening the lengths of sights.

Because of the great variety of conditions encountered, it is obviously impossible to say which of the above sources are of greatest importance for all conditions. However, it may be said that for the ordinary conditions and precision of results, the first four listed are more important than those which follow. Also, from the above discussion it may be noticed that the most important precautions to observe are these: (1) keep the instrument in good adjustment; (2) keep the bubble centered when reading the rod; (3) keep the rod plumb; and (4) keep the foresight and backsight distances equal.

3-26. Mistakes Mistakes which are commonly made in leveling work are the following:

1. *Misreading the Rod.* This refers to reading as one number what in reality is another; e.g., reading the rod as 5.32, when the correct reading is 5.37. Or, what is most dangerous on long lines of levels, reading the wrong foot mark. This happens most often when the marks on the rod are obscured by leaves on trees, grass at the bottom

of the rod, etc. This kind of mistake is to be prevented by habitually looking at the markings and numbers on both sides of the cross-wire reading.

2. *Recording and Computing.* As in other kinds of work, mistakes may be made in calling numbers and in recording them. Such numbers should always be called back, to avoid mistakes. Also, since level notes require additions and subtractions for all results, there is a constant danger of mistakes in these computations. For this reason the check indicated in Fig. 3-17 is invariably applied.

In many level parties, the rodman, as well as the levelman, keeps and computes the notes for all turning points and benchmarks, thus providing a further check against mistakes.

3-27. Checks The checks to be applied in leveling work are of two kinds; first, those which apply to the computations in the field notes, and second, those which apply to the field work itself. The principal check to the field notes is that indicated above.

The field work is checked by closing all level circuits either upon the initial benchmark or upon another benchmark whose elevation is known. As in the case of checks on the field notes, no important use is made of any results in leveling unless this check has been effected.

3-28. Accuracy Since practically all of the errors in leveling are accidental in their effect, the final error of closure in a level circuit is proportional to the square root of the number of rod readings. Accordingly, assuming that the number of readings per mile will generally be about the same, the accuracy or the size of the maximum permissible error of level work is expressed as some coefficient times the square root of the distance in miles. Values which have been accepted are as follows:

$$0.017\ \text{ft}\ \sqrt{\text{miles}} = \text{First-order accuracy}$$
$$0.035\ \text{ft}\ \sqrt{\text{miles}} = \text{Second-order accuracy}$$
$$0.05\ \text{ft}\ \sqrt{\text{miles}} = \text{Third-order accuracy}$$
$$0.10\ \text{ft}\ \sqrt{\text{miles}} = \text{Fourth-order accuracy}$$

3-29. Specifications Although it would be quite impossible to specify the procedure under all conditions to attain a given order of accuracy, it may be stated that under the usual field conditions and in regard to the principal sources of error, the accuracy of 0.1 ft

$\sqrt{\text{miles}}$, which is suitable for many engineering projects, will be secured by conforming to the following specifications:

1. The instrument to be kept in adjustment so that the line of sight shall not deviate from the axis of the bubble tube by more than 0.02 ft in a distance of 300 ft.
2. The bubble to be centered within $\frac{1}{4}$ division whenever a rod reading is taken.
3. The average error in estimating where the cross-wire cuts the rod not to exceed 0.01 ft. This condition requires that the length of sight shall not ordinarily exceed 300 ft. The target is not necessary except for unusual difficulty in sighting the rod.
4. The rod to be plumbed carefully for all readings.
5. Backsight and foresight distances to be equalized by estimation and ordinary care to be exercised with respect to all other sources of error.

3-30. Availability of Vertical Control Data The basic vertical control system for the nation consists of an extensive network of lines of first- and second-order differential leveling executed by the U.S. Coast and Geodetic Survey. Since 1878 a total of over 450,000 miles of levels of these accuracies have been executed. In addition, many more thousands of miles of third-order levels have been run between the first- and second-order lines. In a single year, 1959, the U.S. Geological Survey alone extended approximately 15,000 miles of third-order levels.

Vertical control data are available upon request from numerous federal agencies, such as the U.S. Coast and Geodetic Survey, U.S. Geological Survey, U.S. Corps of Engineers, and the Tennessee Valley Authority. In addition, there are state agencies like the Department of Public Works of the Commonwealth of Massachusetts, and the Maryland Bureau of Control Surveys and Maps from which benchmark elevations can be obtained.

Occasionally the engineer will encounter, quite by accident, a standard benchmark and desire its elevation to control or check leveling operations. It cannot be too strongly recommended that the name of the organization on the tablet be carefully noted, as well as all the stamped numerals and identifying letters on the disc. A direct inquiry to this organization will result in the receipt of the latest, adjusted elevation on a specific datum, usually the 1929 mean sea-level datum.

A typical benchmark tablet is shown in Fig. 3-21.

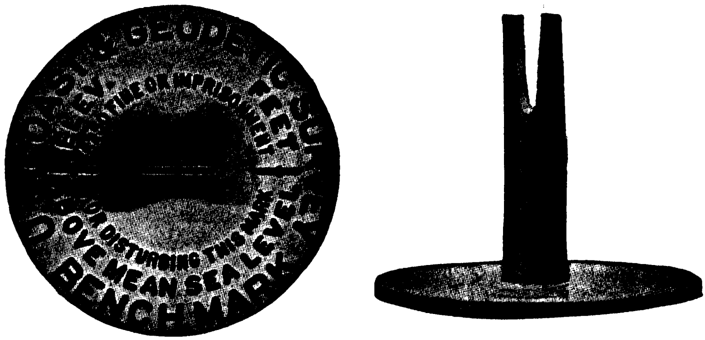


FIG. 3-21. Benchmark Tablet.

3-31. Reciprocal Leveling Where a line of levels crosses a wide and deep ravine, or a river, it is convenient and sometimes necessary to take sights much longer than is ordinarily permissible. For such sights the errors of reading the rod, the curvature of the earth, and the nonadjustment of the instrument become important, and special methods are employed to minimize their effects.

The error in reading the rod is reduced by using a target and taking the mean of a number of readings. The errors due to the nonadjustment of the instrument and the curvature of the earth are eliminated by a special method called "reciprocal leveling," which will now be described.

The procedure is as follows (Fig. 3-22): the instrument is set up a

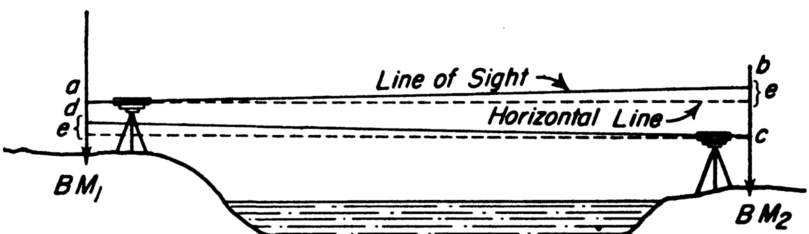


FIG. 3-22. Reciprocal Leveling.

short distance from B.M.$_1$ and readings a and b are taken. Obviously the near reading a is without error and the far reading b is subject to the unknown error e due, it may be assumed, to the nonadjustment of the level. The instrument is then set up near B.M.$_2$ and readings

c and d are taken. Here, the reading c is without error and d is subject to the error e. For the readings taken at B.M.$_1$ the true difference in elevation, D, between B.M.$_1$ and B.M.$_2$ is given by the equation $D = (b - e) - a$; also for the readings taken at B.M.$_2$, $D = c - (d - e)$. Adding these equations, we have

$$D = \frac{(b - a) + (c - d)}{2} \tag{3-4}$$

from which the unknown error e has been eliminated. Since this error may be due to either or both sources mentioned above, the result will be free from errors of both kinds.

If it is plainly evident which benchmark is the higher, the matter of the signs of the rod readings should cause no difficulty. But, if it is doubtful which benchmark is the higher, it will be necessary to pay due regard to the signs of the readings. Since the elevation of B.M.$_1$ is known (or may be assumed), the readings on it may be called backsights; and the readings on B.M.$_2$ will then be foresights. Evidently, the above equation may be written

$$D = \frac{(a + d) - (b + c)}{2} \tag{3-5}$$

and if each backsight is given a plus sign and each foresight a minus sign, then the true difference in elevation will be,

$$D = \frac{(\text{sum of backsights}) - (\text{sum of foresights})}{2} \tag{3-6}$$

If the sign of the right-hand member of this equation is plus, then B.M.$_2$ is higher than B.M.$_1$, and vice versa.

It may be noted that for the conditions shown in Fig. 3-22 the sign of the equation is minus and, therefore, B.M.$_2$ is lower than B.M.$_1$

3-32. Profile Leveling For engineering purposes, a ground profile is the trace of the intersection of an imaginary vertical surface with the ground surface. This profile is usually plotted on specially prepared profile paper, on which the vertical scale is much larger than the horizontal; and on this plotted profile, various studies relating to the fixing of grades and the estimating of costs are made. The field work of profile leveling provides the data for this work.

Assuming that the alinement has been fixed on the ground by setting stakes at 100-ft stations, the level party first determines, by the usual procedure of differential leveling, the height of the instru-

ment which has been set up conveniently near the line of stakes. Foresight readings, called "ground rod" readings, are then taken on the ground at each stake and at intermediate "plus-station" points where there is a marked change in the ground slope.

Since these ground-rod readings are used for plotting only and have no relation to the determination of benchmark elevations, they are taken to the nearest *tenth of a foot* only. Accordingly, all elevations of ground stations are computed to the nearest tenth of a foot only.

A staked center line, profile and form of notes are shown in Fig. 3-23, *a*, *b*, and *c*.

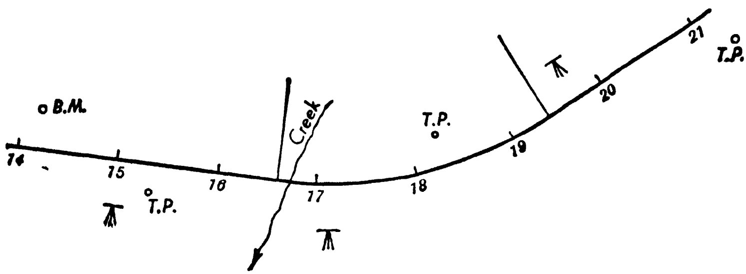
Fig. 3-23*a* shows that the location of the instrument and turning-point stations are independent of the center line. Fig. 3-23*b* shows the plotted profile of the ground line, the horizontal and vertical scales being 100 ft and 20 ft per inch, respectively. From the shape of the ground profile it is apparent why it was necessary to take plus-station readings to show the stream crossing and the changes in slope.

Fig. 3-23*c* shows that the profile-level notes are similar to those for differential levels except that a separate column, headed G.R., is used for the ground-rod readings; and that several of these readings, depending on field conditions, are recorded between turning points.

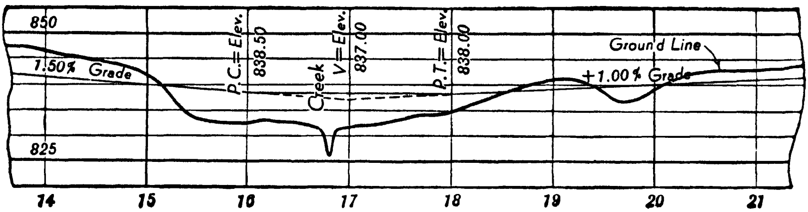
It may be added that when the levelman is reading the rod to the nearest tenth of a foot only, he need not use the extreme care in centering the bubble that is necessary when sighting on benchmarks or turning points.

3-33. Cross-Section Leveling Cross sections are profiles taken transverse to the centerline of a project. They provide the data for estimating quantities of earthwork and for other purposes. Two classes of cross sections may be mentioned: (1) roadway cross sections and (2) borrow-pit cross sections.

Roadway Cross Sections.—The procedure consists of taking profile readings at right angles to the centerline at each station along the route and at any plus-stations where a cross section is necessary, correctly to represent the ground surface. For a roadway, the cross-section profile extends both to the right and to the left of the centerline, usually to the right-of-way fence, or at least, as far as any possible earthwork will be constructed.



(*a*) Plan



(*b*) Profile

HIGHWAY LOCATION-FONTANA TO WALWORTH						Dietzgen Level No. 22 June 15, 19	F. R. Grant ⊼ M. Hughes Rod 10
Sta.	B.S.	H.I.	F.S.	G.R.	Elev.		
B.M.	1.76	849.63			847.87	Spike in Pole 60' Left of Sta. 14 + 20	
14				2.2	847.4		
15				7.0	842.6		
T.P.	0.63	838.66	11.60		838.03		
+40				4.9	833.8		
16				6.3	832.4		
+70				6.8	831.9	Spring Creek	
+75				13.2	825.5		
+80				6.6	832.1		
17				6.2	832.5		
18				4.5	834.2		
T.P.	9.76	846.32	2.10		836.56		
19				5.1	841.2		
+40				6.2	840.1		
+60				10.1	836.2		
20				8.5	837.8		
+40				3.5	842.8		
21				3.7	842.6		
T.P.			3.24		843.08	Root of 10-in. Oak 40' Right of 21 + 20	
	12.15		16.94		847.37		
			12.15	Check	4.79		
			4.79				

(*c*) Field Notes

FIG. 3-23. Profile Leveling.

CROSS-SECTION NOTES						Route 47			November 4, 19	
Sta.	B.S.	H.I.	F.S.	Elev.		Berger Level No. 4 Fair, Cool 10 in. Oak, 40'L. Sta. 412			Williamson, Inst. Lessler, Rod Rogers, Tape	
B.M.	6.15	757.58		751.43						
	7.32	761.74	3.16	754.42		Left		℄	Right	
						30 ft	12 ft.		12 ft	30 ft
415						4.6/57.1	5.0/56.7	5.8/55.9	6.2/55.5	6.8/54.9
+50						6.5/55.2	7.1/54.6	7.2/54.5	7.6/54.1	8.2/53.5
416						4.0/57.7	4.2/57.5	4.5/57.2	4.8/56.9	5.4/56.3
417						2.8/58.9	3.0/58.7	3.1/58.6	3.6/58.1	4.0/57.7
T.P.			4.10	757.64						

FIG. 3-24. Field Notes for Cross-Section Leveling.

The notes are kept somewhat as shown in Fig. 3-24 for which it is assumed that readings extend 30 ft to the right and to the left of the centerline. Each ground-rod reading is recorded on the right-hand page as the numerator of a fraction; the denominator is the elevation of the point, found by subtracting the numerator from the H.I. The differential levels for determining the heights of instrument are recorded as usual on the left-hand page.

Fig. 3-25 shows the manner in which the cross-section data are plotted and also the relation of the proposed roadway to the ground line. The same scales are usually used both vertically and horizontally.

From such cross sections the areas may be calculated, or determined by planimeter, and volumes computed as described in Art. 7-17.

Borrow-pit Cross Sections.—For borrow pits, for some contour maps, and for some other purposes, it is desirable to take ground-rod readings at the corners of squares of regular dimensions as shown in Fig. 7-7.

In this case, the rod readings are usually designated by some simple system of rectangular coordinates and recorded as are profile ground-rod readings. For example, starting from some assumed origin, the abscissas of points may be designated by numbers 1, 2, 3, etc., and ordinates by letters a, b, c, etc. Accordingly, the ground-rod readings may be designated as a-1, a-2, b-1, b-2, etc.

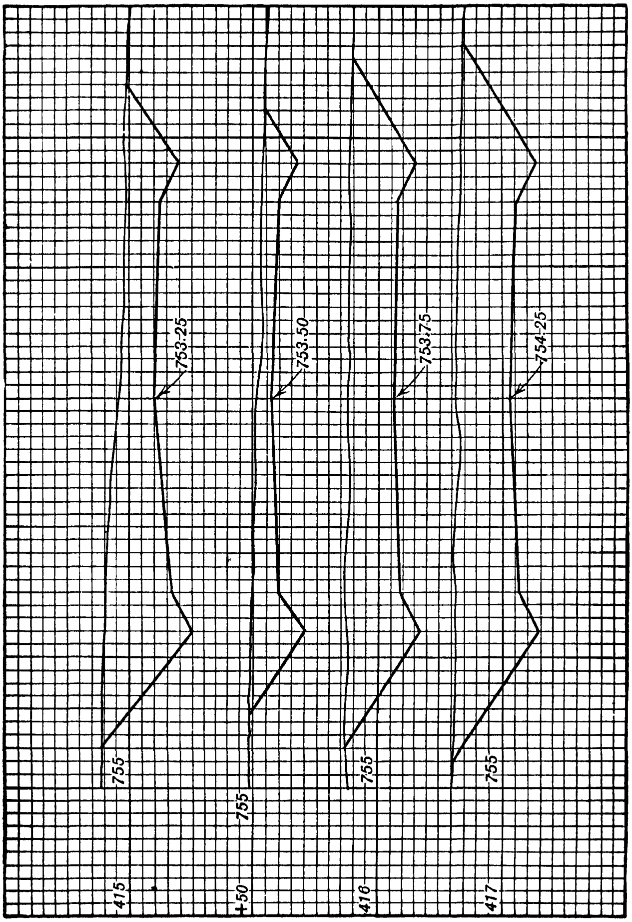


FIG. 3-25. Plotted Cross Sections.

3-34. Trigonometric Leveling The determination of differences of elevation from observed vertical angles and either horizontal or slope distances is termed *trigonometric leveling.*

Fig. 3-26 depicts the usual situation: the sight distance is nominal,

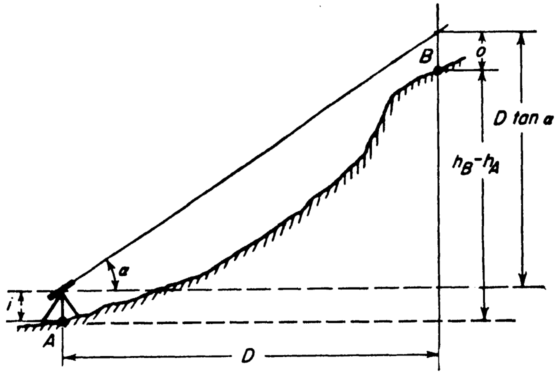


Fig. 3-26. Trigonometric Leveling—Short Sight.

the horizontal distance, D, is known, and the vertical angle, α, has been measured with an engineer's transit as described in Art. 5-9. The height of the telescope above point A is denoted by i, and the vertical angle is read to a point which is situated the vertical distance o above station B.

The difference of elevation is given by the expression:

$$h_B - h_A = D \tan \alpha + i - o \tag{3-7}$$

When sights longer than, say, 1500 ft are taken, it is advisable to incorporate the effects of curvature and refraction in the calculation of vertical height.

Fig. 3-27 portrays the essential conditions. The separate effects of refraction and curvature are denoted by h_r and h_c. The net effect of both phenomena can be calculated by the usual expression for C & R (Eq. 3-1) and included in Eq. 3-7 to produce

$$h_B - h_A = D \tan \alpha + 0.021S^2 + i - o \tag{3-8}$$

Note that whereas for situations involving an angle of elevation (upward sight) the sign of the term for C & R is positive, the opposite would be true for an angle of depression (sight downwards from the horizontal).

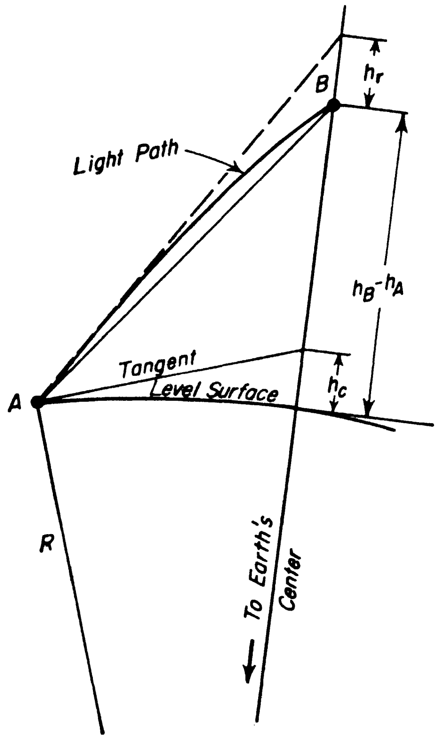
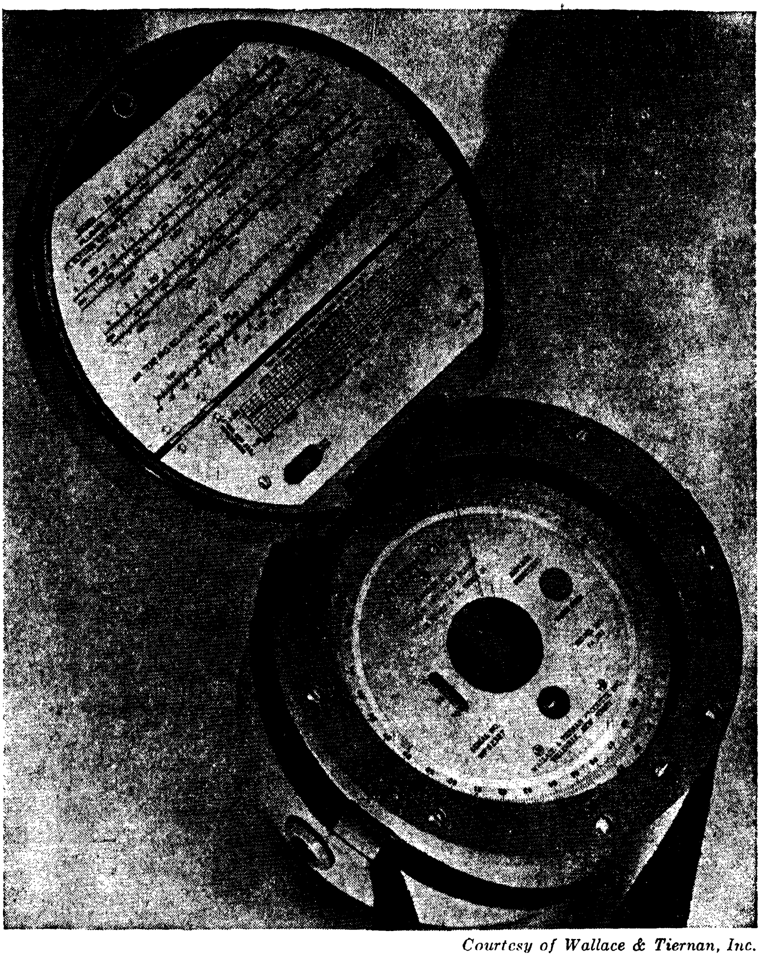


FIG. 3-27. Trigonometric Leveling—Long Sight.

3-35. Stadia Leveling A form of trigonometric leveling, called stadia leveling, is frequently employed in certain mapping work. Inclined distances rather than horizontal distances are utilized together with vertical angles to obtain the difference of elevation. Stadia leveling is explained in Art. 9-7.

3-36. Barometric Leveling The practice of *barometric leveling,* or the determination of elevation by observations of atmospheric pressure, depends upon the basic principle that the pressure caused by the weight of the column of air above the observer decreases as the observer rises in altitude. However, the relationship between pressure and altitude is not constant because air is compressible. Additional factors, although not quite as important, which influence the density of air are temperature and humidity.

The precise *surveying altimeter* (Fig. 3-28) is an improved ver-



Courtesy of Wallace & Tiernan, Inc.

FIG. 3-28. Surveying Altimeter.

sion of the old aneroid barometer. It is remarkably sensitive to changes in atmospheric pressure and its operation is simple. The graduation of the dial into feet makes possible direct readings of altitude. The surveying-altimeter is used primarily to determine differences in elevation between points, one of which is a benchmark or station of known elevation.

Since altimeter surveying is dependent on the measurement of air pressure, all those conditions which affect air density, other than

change of elevation, must be considered if the most satisfactory results are to be obtained. In other words, the assumed pressure-altitude relationship prevails only under certain standard conditions. If the survey is performed when these conditions do not exist, corrections must be applied to the observed differences of elevation.

Altimeter surveys can be conducted in several ways. However, only one procedure, the *single-base method,* will be explained. Two altimeters are employed with the single-base method. One altimeter remains at a point of known elevation where readings of the altimeter and a thermometer are made at regular intervals. The other altimeter, called the roving altimeter, is transported to those points where elevations are desired. Readings are made of the altimeter and a thermometer. To the observed difference of elevation is applied a correction for temperature (and sometimes humidity) and the corrected difference of elevation is combined with the elevation of the base station to give the elevation of the field station.

Barometric leveling is performed when expedient for exploratory surveys and in situations where the accuracy requirements are greatly reduced.

Office Problems

3-1. In leveling across a river the following conditions are given—elevation of $B.M._1$: 832.47 ft; B.S.: 9.24 at a distance of 100 ft; F.S.: 3.28 on $B.M._2$ at a distance of 1200 ft. What is the elevation of $B.M._2$ corrected for the curvature of the earth and refraction? *Ans.* 838.46 ft.

3-2. The elevation of $B.M._1$ is known to be 1745.26 ft. The elevation of $B.M._2$ has been found to be 2069.71 ft with a 12-ft level rod which, by test, was found to be 0.01 ft too long, the error being distributed over the whole length of the rod. Find the corrected elevation of $B.M._2$. *Ans.* 2069.98 ft.

3-3. In testing an engineer's level the following readings were taken: With the instrument midway between stakes A and B, the reading on A, 3.84, and the reading on B, 6.22ft. With the instrument at A, the reading on A was 4.16. What is the computed correct reading on B? *Ans.* 6.54 ft. What is the amount and direction of the inclination of the line of sight if the reading from A on B is 6.58 ft? *Ans.* Upward 0.04 ft.

3-4. In leveling across a river the following conditions are given—elevation of $B.M._1$: 647.35 ft; B.S.: 5.22 at a distance of 200 ft; F.S.: 8.32 on $B.M._2$ at a distance of 1500 ft. Find the elevation of $B.M._2$ corrected for curvature of the earth and refraction.

3-5. The elevation of $B.M._2$ was found to be 516.42 ft by a line of

levels run from B.M.$_1$, whose elevation is known to be 1175.60 ft. A test of the level rod, which was 12 ft long, showed it to be 0.01 ft too short, the error being distributed over the whole length of the rod. Find the corrected elevation of B.M.$_2$ due to this error in the length of the rod.

3-6. Assume that the line of levels of Prob. 3-5 was continued to close on the initial benchmark. What correction should be applied to this elevation on account of the incorrect length of the rod?

3-7. A rod reading of 12.00 ft is taken when the rod at that point was 6 in. out of plumb. (a) What is the amount of the resulting error? (b) If the rod were 1 ft out of plumb?

3-8. An engineer's level is tested with results as follows: With instrument midway between stakes A and B, reading on A, 7.22; reading on B, 5.16. With instrument at A, the reading on A, 4.84, and reading on B, 2.72. (a) Is the line of sight inclined upward or downward? (b) How much? (c) How is the adjustment made?

3-9. The sensitiveness of a bubble tube has been tested with results as follows: With the bubble in one position in its tube the reading on the rod is 5.16. The bubble is then moved 5 divisions and the reading then is 5.28. The bubble divisions are 0.1 in. and the rod is held at a distance of 300 ft. (a) What arc (in seconds) is subtended by one division of the bubble tube? (b) What is the length of the radius of curvature of the bubble tube?

3-10. A line of levels is run from benchmark A in a valley to benchmark B on an adjacent summit. What is the nature of the effect on the determined elevation of benchmark B on account of the following sources of error? (a) Line of sight inclined upward with respect to the axis of the bubble tube. (b) The bottom of the rod worn off. (c) Rod not plumb. (d) Curvature of earth.

3-11. In running a line of differential levels from B.M.$_1$ to B.M.$_2$, the following readings were taken in the order given. Record these in good notebook form, find the elevation of B.M.$_2$, and show the customary check on the computations. B.M.$_1$: elevation 1463.28; rod readings: 11.16, 4.28, 10.32, 1.27, 8.63, 1.77, 0.56, 10.33, 9.17, 2.45, 9.03, 1.80.

3-12. In leveling across a river, reciprocal level readings were taken between two benchmarks A and B as follows: instrument near A, rod on A, 4.21, rod on B, 2.21; instrument near B, rod on B, 4.85, rod on A, 7.03. The elevation of A was 2465.73 ft. Find the elevation of benchmark B. *Ans.* 2467.82 ft.

3-13. If the distance between the benchmarks A and B of Prob. 3-12 is 1600 ft, what is the error in the adjustment of the instrument? *Ans.* The line of sight is inclined upward 0.04 ft in the distance between the benchmarks.

3-14. In running a line of profile levels the following readings were taken in the order given. Assume suitable station numbers and re-

cord the data in good notebook form. $B.M._1$: elev. 934.47; 7.35, 10.1, 7.4, 6.3, 5.8, 2.3, 2.41, 8.67, 9.3, 7.4, 6.2, 5.5, 4.03.

3-15. In carrying a line of levels across a river, the following data are given (let A and B represent the two benchmarks, one on each side of the river): instrument near A, rod on A: 4.14, rod on B (mean of a series): 7.22; instrument near B, rod on B: 5.13, rod on A (mean): 2.35.

If the elevation of benchmark A is 1214.65, find the elevation of benchmark B.

Field Problem 3-1. Differential Levels

Procedure.—Run a line of differential levels, either between two assigned benchmarks, or to close on the initial benchmark. Observe the following precautions; (1) remove parallax in the telescope; (2) center the bubble closely and quickly over each pair of leveling screws, but do not waste time with extreme care in this preliminary step. It is essential that the bubble be centered accurately only when reading the rod; (3) look at the bubble both before and after reading the rod, to make sure it was centered when the reading was taken; (4) keep the backsights and foresights as nearly equal as conditions will permit; (5) keep the rod plumb, and when it is extended, wave it; (6) enter all readings and make all computations as the work proceeds.

Describe all benchmarks carefully and completely, and show the check on all computations.

Field Problem 3-2. Profile Levels

Procedure.—Set numbered stakes at 50- or 100-ft intervals on the proposed line. Begin the levels at an assigned benchmark and close on another, or on the initial benchmark. Readings on the ground are taken to the nearest tenth of a foot only, and resulting elevations are computed to tenths only. Plus readings are taken at any notable changes in slope on the line between the stakes; the distance to such points being paced, or estimated, by the rodman and called to the levelman. The levelman should not waste time by using undue care on ground readings, but all turning points should be solid objects and the readings on these points taken with the same care as in differential leveling. Show the check on the computations.

CHAPTER **4**

ANGLES AND DIRECTIONS

4-1. Remarks Before undertaking a discussion of the transit and its use, there are a few important considerations regarding angles and directions which should be understood. In this connection some reference to the compass is necessary. This instrument was formerly much used in land surveying and for other purposes. It still has a limited use in forest ranging, exploratory surveys, and in mapping; but for construction and land surveying at the present time the compass, as a separate instrument, is obsolete. However, on surveys of any considerable extent in which angles are measured with the transit, the compass as a part of the transit instrument is still used, but only as a check on the transit work. Accordingly, in this chapter, there will be given a few necessary definitions and a brief treatment of the compass and its use.

4-2. Units of Angular Measurement An angle between two lines at a point is given by the difference in the directions of the lines. Only plane angles are considered here. The magnitude of an angle can be expressed in different units, all of which are basically derived from the division of the circumference of a circle in various ways. The principal systems of units are as follows:

1. *Sexagesimal System.* The circumference is divided into 360 parts. The basic unit is the degree (°), which is further subdivided into 60 minutes (60′), and the minute is subdivided into 60 seconds (60″). This system is used exclusively in surveying practice in the United States.
2. *Centesimal System.* The circumference is divided into 400 parts called *grads*. Hence $100^g = 90°$. The grad is divided into 100 centesimal minutes (100^c) and a centesimal minute is divided into

100 centesimal seconds (100^{cc}). The centesimal system has wide usage in Europe.

3. The *mil* is 1/6400 part of a circumference of a circle and will subtend very nearly one linear unit in a distance of 1,000 such units. It is used in military operations.

4. The *radian* is the angle at the center of a circle subtended by an arc having exactly the same length as the radius. One radian equals $\frac{360°}{2\pi}$ or approximately 57.30°. Radians can be employed for certain calculations such as determining the length of circular arcs. The radian, in contrast with the other units of circular measure previously mentioned, is sometimes referred to as the natural unit of angle because there is no arbitrary number, like 360, in its definition.

4-3. Kinds of Angles Various kinds of angles can be used to express difference of direction. The following types (Figs. 4-1 and 4-2) are all horizontal angles.

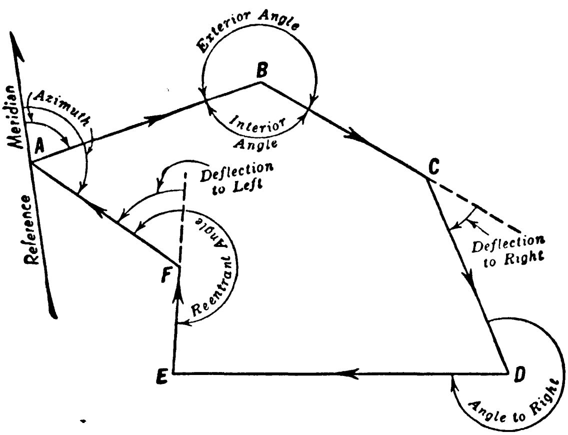


FIG. 4-1. Closed Traverse.

An *interior* angle is one enclosed by the other sides of a closed figure, as shown at B.

An *exterior* angle is one not enclosed by the other sides of a closed figure, as shown at B.

A *deflection angle* is that angle which any line, as CD, makes with the preceding line, BC, produced. A deflection angle may be turned

to the right or to the left. Hence, it is always necessary to indicate by the letters R or L in which direction the angle has been turned.

An *angle to right* is the clockwise angle at any vertex, D, between the back line DC, and the forward line DE.

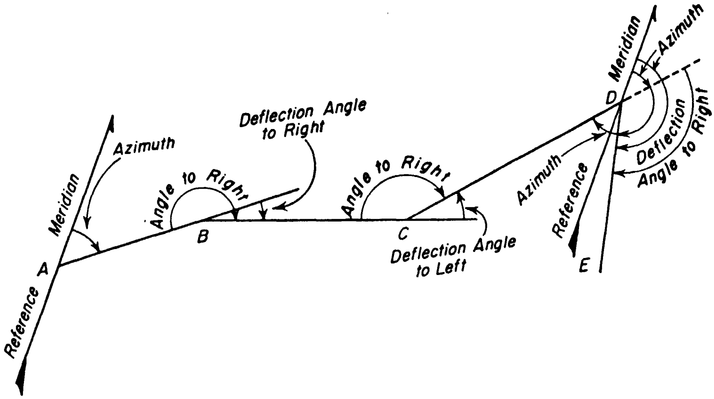


FIG. 4-2. Open Traverse.

A *re-entrant angle* is an interior angle which is greater than 180°, as at F.

An *explement* is the difference between any angle and 360°. Thus the exterior angle is the explement of the interior angle.

4-4. Bearings and Azimuths It is frequently convenient to choose or fix a reference line to which the directions of all the lines of a survey are referred. Such a reference line is called a *meridian,* of which there are four kinds: *magnetic, true, grid,* and *assumed.*

A magnetic meridian is the direction in the horizontal plane taken by a magnetized needle when it comes to rest in the earth's magnetic field.

A true meridian is that meridian through a given point joining the north and south poles of the earth's axis.

A grid meridian is a line parallel to the central meridian or "Y" axis of a system of plane-rectangular coordinates.

An assumed meridian is a direction chosen by considerations of convenience for any particular survey or locality.

The acute angle which a line makes with a reference meridian is called its *bearing,* and since for any given line there is more than one such angle, this term must be further defined.

It is customary to refer directions with respect to the magnetic meridian to both the north and the south ends of the meridian and also both to the west and the east. Thus, in Fig. 4-3 the magnetic bearing of OA is given as N 65° E, and of OB as S 55° E, and accordingly the magnitude of a bearing is never greater than 90°.

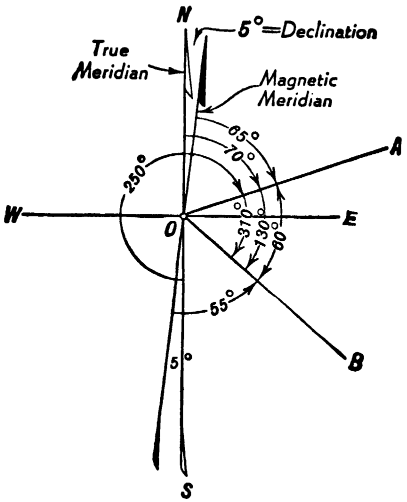


FIG. 4-3. Bearings.

In the figure it is assumed that the magnetic meridian makes an angle of 5° with the true meridian, and this angle is called the *declination* of the needle. It is evident, then, that the true bearing of line OA is N 70° E, but a more common way of designating direction with respect to the true meridian is simply to give the value of the clockwise angle which the line makes with the north end of the meridian, and this angle is called the line's *azimuth*. Thus, the azimuth of the line OA is 70°, and there is no need for the use of letters referring to points of the compass. The magnitudes of azimuths vary from 0° to 360°, hence the azimuth of OB is 130°. Sometimes azimuths are referred to the south end of the meridian, in which case the azimuths of OA and OB are 250° and 310°, respectively. But for any given survey, azimuths are referred to but one end of the meridian only. *In this book, unless otherwise stated, the term azimuth will refer to the north end of the meridian.*

From the above discussion we have the following definitions:

The *declination* of the needle is the angle it makes with the true meridian.

The *magnetic bearing* of a line is the acute angle which a line makes with the magnetic meridian.

The *grid bearing* of a line is the acute angle which the line makes with the grid meridian.

The *true bearing* of a line is the acute angle which the line makes with the true meridian.

The *azimuth* of a line is the clockwise angle which a line makes with the north end of the true, or an assumed, meridian.

Every line has two directions, differing from each other by 180°, and depending on the point of view of the observer, ie., at which end of the line he is stationed. Thus the magnetic bearing of OA, with the observer at O, is N 65° E; but at A, the bearing is S 65° W. This value is termed the back bearing of OA. Likewise, the azimuth of OB is 130° and its back azimuth is $130° + 180° = 310°$.

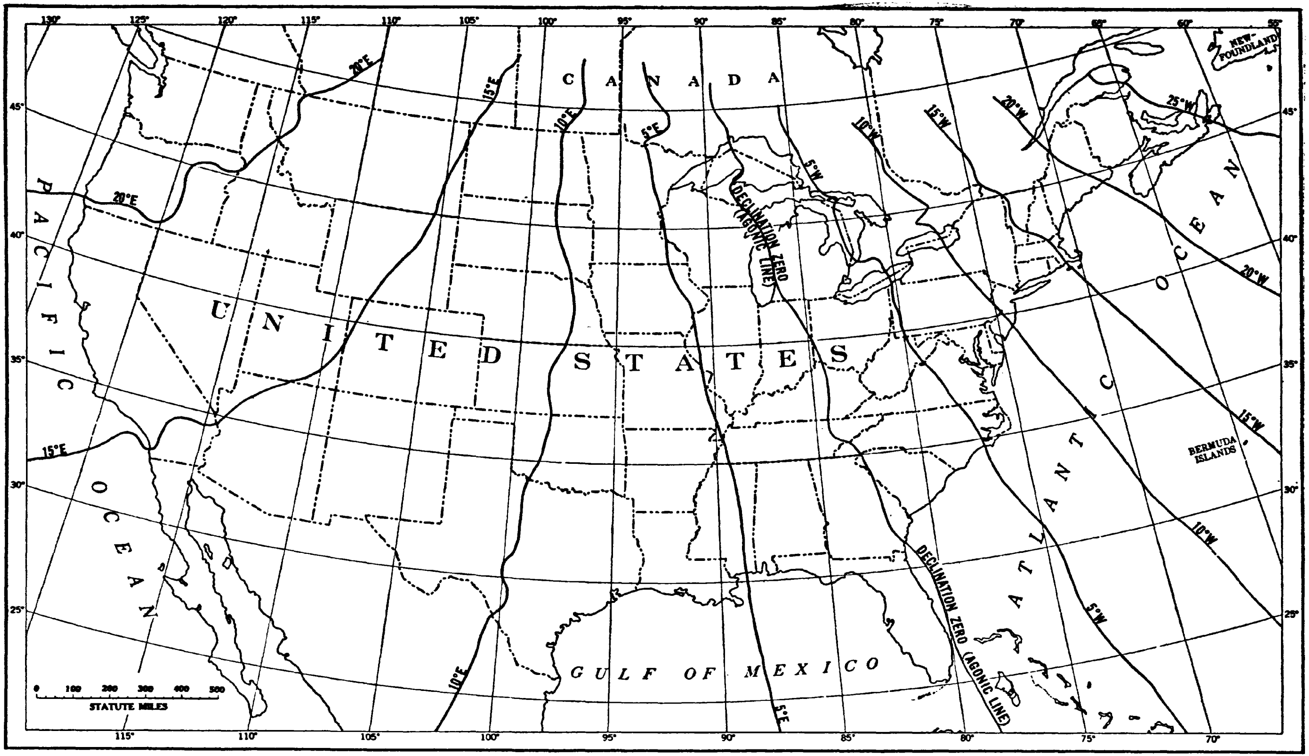
When a meridian is assumed, it is used as though it were a true meridian, and directions are referred to it either as bearings or azimuths, depending on the purpose of the survey.

4-5. The Earth's Magnetic Field The earth's magnetic field is characterized by lines of force which, except for small variations, remain fairly constant in direction. Accordingly, at any given place, the compass needle will indicate the same direction over a considerable period of time. For most places this direction will not be true north but will be either to the east or west of north, depending on the locality. This angle between the direction of the needle and true north has already been defined as the declination of the needle.

Figure 4-4 is an *isogonic chart* of the United States for 1960, which shows lines of equal magnetic declination. The position of the *agonic* line or line of 0° declination is to be noted. To the west of it the declination is east and to the east of it the declination is west.

The magnetic lines of force are also inclined to the horizontal, so that a needle which is balanced before it is magnetized will afterward, in the Northern Hemisphere, have its north end deflected downward. This deflection is called the *dip* of the needle.

4-6. Magnetic Compass The essential features of the compass are shown in Fig. 4-5. These include a circle graduated in quadrants from 0° to 90° both east and west from both the north and south points of the compass; a magnetized steel needle, supported on a steel pivot and jeweled bearing and counterbalanced against the dip of the needle; also a line of sight fixed with respect to the north-south points of the compass box. Since the needle dips down to the north, the counterbalance is placed south of the pivot and serves to indicate the north and the south ends of the needle. It will be noticed that, in sighting any point, since the graduated circle turns while the needle remains stationary on its pivot, the west and east



Courtesy of U.S. Coast and Geodetic Survey

FIG. 4-4. Distribution of magnetic declination in the United States for 1960.

points of the circle are interchanged, thus to give the correct bearing if the north end of the needle is read.

A clamp is provided to lift the needle from its pivot when not in use. It is important that this be done; otherwise, the needle will jar on its bearing and soon become sluggish and insensitive.

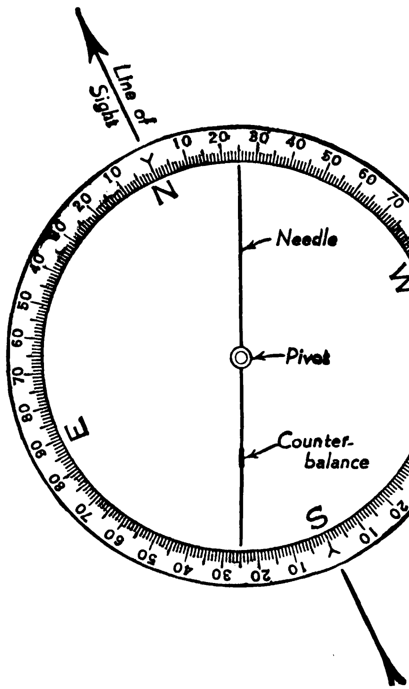


FIG. 4-5. Magnetic Compass.

As a part of the transit, the compass box is mounted on the upper plate of the transit between the standards. See Fig. 5-1. The telescope, of course, provides the line of sight.

4-7. Reading a Bearing To read a bearing with the compass, the transit plate is leveled, and the needle is released by its clamp and allowed to come to rest. The telescope is sighted along the line whose bearing is desired and the north end of the needle then indicates the bearing. If the south end of the needle were read it is obvious that the back bearing of a line would be read. In Fig. 4-5 the bearing of the line of sight is found to be N 25° W.

It will later be found that the transit telescope may, without turning the plate or compass box, be sighted in the reversed direction. Now the line of sight is said to be reversed and the correct (forward) bearing for this direction of the line of sight will be obtained by reading the south end of the needle, or S 25° E. The same result is also obtained if the north end of the needle is read and the letters indicated by the compass box reversed.

4-8. Calculating Bearings or Angles An inspection of Fig. 4-3 will show that, if the bearings of two lines are given, the angle between them may be computed; also, if the bearing of one line is given and the angle which a second line makes with it is also given, then the bearing of the second line may be found. A bearing thus found is termed a *calculated* bearing.

For example, the magnetic bearings of OA and OB are N 65° E, and S 55° E, from which it is evident that angle $AOB = 60°$. Likewise, if the true bearing of OA is N 70° E and angle $AOB = 60°$, then the calculated bearing of OB is S 50° E.

Similar relations obtain with respect to the azimuths of lines and the angles between them.

It is sometimes desirable to change magnetic bearings into true bearings and vice versa. If the declination of the needle is shown, this can readily be done by drawing the figure to show the proper relations, after which the computations are easily made. Thus, in Fig. 4-3, if the magnetic bearing of OA is N 65° E and the declination of the needle is 5° E, then the true bearing of OA is N 70° E. Also, if the true bearing of OB is S 50° E, the magnetic bearing is S 55° E.

4-9. Changes in the Declination The direction of the magnetic meridian at any given place is subject to a number of variations that constitute a source of error which may or may not be serious, depending on the field conditions and the accuracy desired.

There is a *daily* variation, i.e., a slight swing of the needle of a few minutes of arc between morning and afternoon observations; there is also a small *annual* variation from month to month. These are quite negligible for ordinary compass work.

A *secular* variation, extending over a long period of time (about 250 years) is of greater importance because, while the variation

from one year to the next is slight, it is a systematic movement which, in a term of years, attains considerable magnitude. By reason of this variation the direction of the magnetic meridian, at any given place, changes slowly (perhaps 3′ per year) to the west for a long period of time and then it swings back to the east over a similar period. At the present time the swing is toward the west and, if the declination of the needle at any given place were 5° east of north 30 years ago, then, assuming a rate 3′ westward per year, it would be 3°30′ east now, and it will be 2° east, 30 years hence. This is a matter of some importance in retracing compass surveys made many years ago.

4-10. Mistakes *Misreading the Quadrant Letters.*—A common mistake is to misread the quadrant letters when taking a bearing near the cardinal points of the compass. Thus, a bearing of S 1°30′ E is misread as S 1°30′ W, or a bearing of S 89°30′ W is misread as N 89°30′ W, etc.

Transposing Quadrant Letters.—Frequently the quadrant letters are transposed either in reading the bearing or in recording it. Thus, a bearing of N 10° W, is misread or recorded as S 10° E. This mistake always results if the wrong end of the needle is observed.

Misreading the Circle.—Mistakes are easily made in reading the circle. For example, a bearing of S 39°30′ W is misread as S 41°30′ W.

4-11. Accuracy From what has been stated regarding the nature and size of the errors in compass work, it may be said that the average error in determining directions with the kind of compass supplied on transit instruments varies from 5′ to 10′ or more. Most of the sources of error are accidental, however, and on this account, the errors of closure on long compass surveys are often surprisingly small.

Because of the many conditions affecting compass work, it is impossible to give an accurate quantitative statement regarding the accuracy of such work that would be generally applicable; but it will not be far from the truth to say that the error of closure (see Art. 7-7) attained on compass surveys will usually be in the neighborhood of 1/500 to 1/1000. Such accuracy is suitable for comparatively rough estimates of timber or drainage areas, for mapping purposes,

etc. However, because of the limited number of surveys to which the compass is adapted, and for other reasons, its use, as a separate instrument, is practically negligible.

Office Problems

It is helpful in solving problems involving bearings and azimuths, to draw a reasonably accurate figure and draw the cardinal direction lines through each vertex.

4-1. Given the magnetic bearing AB = S 42° E and the magnetic declination 4°30′ E. Find the true bearing of AB. *Ans.* S 37°30′ E.

4-2. Given the bearings OA = N 38°00′ E and OB = S 31°15′ E. Find angle AOB. *Ans.* 110°45′.

4-3. Given the bearing OE = N 22°15′ W and the clockwise angle EOF = 130°45′. Find the bearing of OF. *Ans.* S 71°30′ E.

4-4. AB is one side of an equilateral triangle and has a bearing of N 70° E. Vertex C lies south of line AB.

(a) Find the bearings of sides BC and CA.

(b) Find the forward and back azimuths of the three sides.

Ans. (a) BC = S 10° W; CA = N 50° W.

(b)

	Forward Azimuth	Back Azimuth
AB	70°	250°
BC	190°	10°
CA	310°	130°

4-5. Given the magnetic bearing AB = N 72° E and the magnetic declination 3° W. Find the true bearing of AB.

4-6. Given the bearings OA = N 62°15′ E and OB = N 81°30′ W. Find the angle AOB.

4-7. Given the bearing OC = S 10°14′ W and the clockwise angle COD = 83°17′. Find the calculated bearing of OD.

4-8. At a given place in 1875 the magnetic bearing of a line was N 89°15′ W, and the declination of the needle 5° W. At the present time the declination is 2°30′ W.

(a) What is the present magnetic bearing of the line?

(b) What is the true bearing of the line?

4-9. The magnetic bearings of the sides of a field have been observed as follows: AB = S 25°30′ E; BC = S 12°00′ W; CD = S 68°15′ W; DA = N 18°45′E. Find the interior angles.

4-10. If the declination of the needle is assumed to be 4°30′ E for the bearings given in Prob. 4-9, (a) find the true bearings, and (b) change the true bearings into azimuths.

4-11. The interior angles of a field are as follows: A = 73°08′; B = 132°22′; C = 88°47′; and D = 65°43′. The magnetic bearing of

AB is N 65°30′ E. If the direction of the courses is taken to be clockwise, what are the calculated bearings of the other sides of the field?

4-12. In a survey, the following magnetic bearings have been observed: AB = N 62°15′ E; BC = S 81°00′ E; CD = N 75°45′ E; DE = S 13°00′ W; and EF = S 0°30′ E. Find the deflection angles.

CHAPTER 5

THE TRANSIT

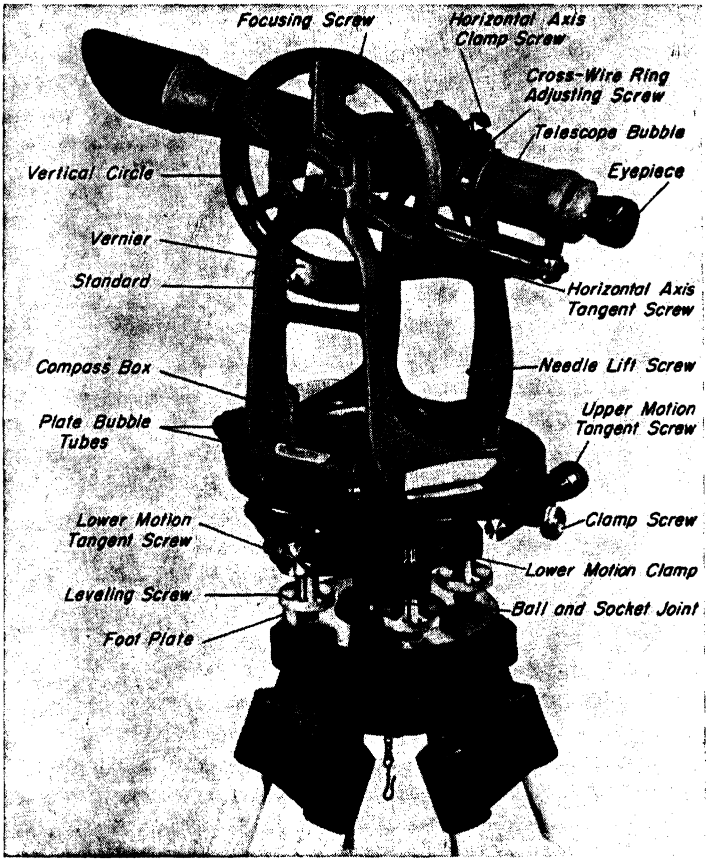
5-1. Remarks The transit is an instrument of precision used principally to measure angles in the horizontal plane and to prolong straight lines. It is usually equipped also for measuring vertical angles, for measuring distances by the stadia method, for differential leveling, and for taking compass bearings. Accordingly, it is sometimes called the "universal" surveying instrument.

Although this chapter is largely concerned with the American engineer's vernier transit, a sufficient introduction is provided to the optical theodolite to acquaint the student with its general design characteristics and field capabilities.

5-2. Description of the Transit A photograph of a transit is shown in Fig. 5-1. The essential parts of a transit are (1) the telescope, mounted on the upper plate, which, in turn, is supported by an inner solid spindle; (2) the graduated circle, being a part of the lower plate and supported on an outer hollow spindle; and (3) a leveling head which provides a means of leveling the instrument.

The outer hollow spindle rotates in a bearing provided by the leveling head and may be clamped to it by means of the lower-motion clamp. The inner spindle rotates inside the outer spindle and may be clamped to it by means of the upper-motion clamp. Since the inner spindle carries the circular verniers on the upper plate and the outer spindle carries the graduated circle on the lower plate, it is possible to rotate the verniers within the graduated circle; or, by the upper clamp, to fix the verniers in any desired position with respect to the circle and then to rotate them, in their fixed position with respect to each other, by means of the outer spindle.

A compass is usually provided, and if so, it is placed on the upper plate between the standards.



Courtesy of Keuffel & Esser Co.

FIG. 5-1. Engineer's Transit.

A plumb bob is attached beneath the leveling head, and a shifting plate, beneath the foot plate, is provided by means of which the transit head, as a whole, can be shifted laterally a small distance, without moving the tripod, thus to bring the plumb bob precisely over a fixed point.

The two castings, by means of which any two parts of the instrument are clamped together, do not bear directly against each other, but one has a lug which bears against a stiff spring on the other. A screw, called a *tangent screw,* opposes the spring and thus a small amount of play, or motion, is provided between the two parts, which

motion is precisely controlled by turning the tangent screw. Each clamp, therefore, has its tangent screw.

Two opposite verniers, 180° apart, called the *A* and *B* verniers, are provided for the horizontal circle.

5-3. Definition of Terms A few terms commonly used in connection with transit work may be defined as follows:

Orientation.—As applied to transit work, this term refers to the fixed position of the horizontal plate with respect to an established line through the instrument station. Thus, the transit is said to be oriented with respect to a given line when, with the line of sight directed along it, the vernier reads a given angle.

Backsight.—A backsight in transit work refers to a sight taken on a point, usually the last preceding, so as properly to orient the transit.

Foresight.—A foresight with the transit is a sight taken to fix the direction of a line.

Normal or Direct Position.—The *normal* or *direct* position of the transit is that in which the eyepiece is above the *A* vernier and the attached bubble tube is below the telescope.

Inverted Position.—The inverted position of a transit is that in which the attached bubble tube is above the telescope and the eyepiece is over the *B* vernier.

To Reverse the Instrument.—To reverse the instrument means to rotate the upper plate, including the standards and telescope, approximately 180° about the vertical axis.

Horizontal Axis.—The horizontal axis, accurately speaking, is the imaginary axis of the trunnion which supports the telescope. However, the trunnion itself is commonly spoken of as the horizontal axis.

Vertical Axis.—The vertical axis is the center line of the inner spindle.

Upper Motion.—That part of the instrument which rotates on the inner spindle, and which includes the upper plate, the standards, and the telescope, is commonly called the *upper motion.* It is controlled by the upper-motion clamp and tangent screw.

Lower Motion.—That part which rotates on the outer spindle and which includes the graduated circle is commonly called the *lower motion.* It is controlled by the lower-motion clamp and tangent screw.

5-4. Care of the Transit The materials and workmanship of a good transit are such that with proper usage it will last a lifetime. On the other hand, and aside from accidents, a half day's careless usage may injure it permanently. Accordingly, careful attention should be given to all that has been said previously in Art. 3-17 about the care of equipment. In particular, the clamps of the transit should receive careful treatment. The beginner is likely to fear that the plates will slip and consequently tighten the clamps too much. The parts of the instrument are perfectly fitted such that a snug bearing is all that is required. When returning a transit to its box, the leveling head should be shifted to a central position and the foot screws evened all around.

To avoid tarnishing, the graduated circles and verniers should not be touched with the fingers. These surfaces gradually become tarnished in use, however, especially the vertical circles, and should then be cleaned. This is best done by applying a thin film of oil which is left for a few hours and then removed with a soft rag. Excessive rubbing of the graduations should be avoided, for this impairs the sharpness of the engraved lines.

5-5. The Vernier The application of the vernier to a circular scale is shown in Fig. 5-2. Two arrangements are shown, the vernier at (*a*) reading to minutes and the one at (*b*) reading to half minutes, or thirty seconds. Each one is a double vernier, i.e., it is arranged to read in either direction, to the right or to the left, depending upon the direction of rotation in measuring the angle.

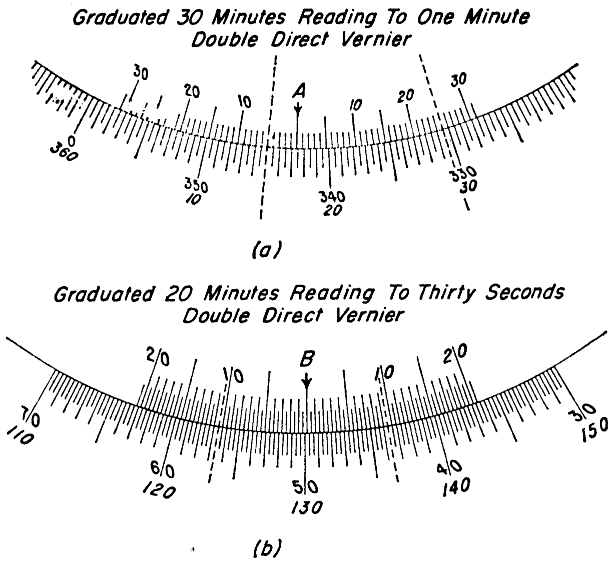
The fact that vernier (*a*) reads to minutes may be determined by Eq. (3-3):

$$D_s = \frac{s}{n} = \frac{30'}{30} = 1'$$

The value of an angle is found by first reading the scale, then the vernier, and then adding the two readings together. Thus, the scale reading (inner circle) is seen to be 342°30′, and looking along the vernier, the fifth vernier division coincides with a scale division, hence the reading is 342°30′ + 5′ = 342°35′. Similarly, reading the outer circle, the scale reading is 17°, and the vernier reading is 25′. It is important to note that the vernier is always read in the same direction as the scale. This relation is indicated by the slope of the numbers, both on the vernier and on the scale. Thus, the numbers

which slope to the left on the scale and on the vernier are to be taken together.

In Fig. 5-2b it is seen that the vernier divisions are numbered to 20, and these numbered divisions are divided in half, thus giving 40



Courtesy of Keuffel & Esser Co.

FIG. 5-2. Transit Verniers.

actual divisions. Accordingly, $D_s = 20'/40 = 30''$; and the two vernier readings are $49°50'30''$ (inner) and $130°09'30''$ (outer), respectively.

Various arrangements of verniers will be found on the different instruments and the transitman should be careful to determine correctly the characteristics of each vernier before it is used.

5-6. Setting Up the Transit Most of what has been said in Art. 3-19 about setting up a level applies also to the transit. When properly set up the instrument should not only be leveled, but also centered over a fixed point. To do this, the tripod legs are adjusted approximately to the ground surface, keeping the foot plate of the leveling head as nearly level as may be convenient. Then the instrument is carried horizontally until it is nearly centered over the point, the tripod legs are pushed into the ground firmly, the leveling

head shifted until the plumb bob is over the point, and the plate is leveled with the foot screws.

5-7. To Measure a Horizontal Angle Before using the transit telescope the eyepiece should be focused on the cross-wires as was explained in Art. 3-6. It has been explained that with each clamp there is provided a tangent screw for the precise control of one part with respect to the other. By this arrangement the line of sight is fixed precisely on any point, first by directing the telescope toward it and bringing the cross-wire close to it, using the hands to turn the instrument on its spindle, and then setting the clamp and using the tangent screw. It should be added that a tangent screw belonging to a certain clamp should be used with that clamp, and no other.

Accordingly, with the instrument properly set up, the procedure of measuring an angle between two points A and B, may be described as follows: set the A vernier to read zero, using the upper clamp and tangent screw; loosen the lower clamp and set the vertical cross-wire on point A, using the lower clamp and tangent screw; loosen the upper clamp and sight point B, using the upper clamp and tangent screw; read the value of the angle with the A vernier.

The B vernier is provided for special purposes to be explained later.

5-8. To Prolong a Straight Line One of the important uses of a transit is that of prolonging a straight line. Three or four methods are available but, for ordinary purposes, the best one is that called the *method of double sights* which will now be explained.

It may be supposed that the two points A and B fix the direction of the line which is to be prolonged. The instrument is set up over B and a backsight is taken on A with the telescope in the normal position. The telescope is inverted and a temporary point C is marked on a stake set on the line of sight and at a distance as far as can conveniently be seen. The instrument is then reversed, and a second sight is taken on A with the telescope remaining in the inverted position. The telescope is then made direct and a second temporary point D is marked on the foresight stake. A point midway between C and D is now found and fixed as a permanent point on the true prolongation of line AB.

If the points C and D should happen to coincide, that is the permanent point sought.

Possibly the two temporary points will not fall on one stake. If not, two or more stakes may be necessary to complete the process, care being taken not to disturb a point which has once been set, and that the final point is properly fixed.

5-9. To Measure a Vertical Angle When the angle of elevation to any object is desired, it is necessary first to determine the direction of a horizontal line and then to measure the vertical angle between this line and the line to the object. The direction of a horizontal line is determined by the bubble attached to the telescope, and the vertical circle serves to measure angles in the vertical plane. Accordingly, a vertical angle is measured as follows: The transit is set up and carefully leveled. The telescope is directed toward the object whose elevation is desired, the telescope bubble is centered, and the vernier is read. For an instrument properly adjusted, this reading is zero, but usually the reading gives a small angle called the *index error,* which should be carefully observed, especially the sign, i.e., whether the error is to be added or subtracted. The elevated object is then sighted and the vernier read. This reading corrected for the index error is the angle desired.

If the vertical arc of the transit is a full circle and if the object is observed first with the telescope normal and then with the telescope inverted, the mean of the two readings will be the correct value of the vertical angle, the index error having thus been eliminated.

5-10. To Measure an Angle by Repetition To measure an angle by repetition, the verniers are set at zero as nearly as may be, with the telescope in the normal position, and the line of sight is set on the initial station. Both *A* and *B* verniers are then read and the upper motion released, turned clockwise to the next station, and the pointing made with the upper motion. The *A* vernier is then read to the nearest minute, and, assuming that six repetitions are to be made, the process is repeated until three repetitions have been completed, when both *A* and *B* verniers are read carefully. This reading, doubled, should agree with the sixth repetition within one minute, or the series should be repeated. After six repetitions have been made with the telescope direct, the telescope is reversed, pointed at the right-hand station with the lower motion, turned clockwise to the left-hand station (through the explement of the angle) and set with the upper motion. This procedure subtracts the value of the angle

each time until, at the last pointing, the verniers should read zero again. It will be noted that throughout this procedure the instrument has been turned clockwise, for the reason that if any creeping of the spindles takes place, it will accumulate and become evident on the final reading. Otherwise, the error would be present in the reading of the repeated angle but would not appear in the reversed procedure when the readings came back to zero.

By this procedure an angle can be repeated as many times as may be desired. But experience has shown that nothing is gained by using more than six repetitions, direct and reversed (called a set) because of other concurrent errors. However, greater accuracy can be obtained by using additional sets of repetitions. In this case, the initial setting of the verniers for each set should be changed by an angle equal to 360° divided by the numbers of sets, so that any errors in the graduations of the circle may be eliminated.

A form of notes showing six repetitions of an angle is shown in Fig. 5-3. It will be noticed that for the case of six repetitions, using

ANGLES BY REPETITION
STATION: DEL.

Tel. D or R	Rep	Angle	Vern. A	Vern. B	Mean of Verns.	Arc	Angle D & R	Mean Angle
D		0°00'	10"	10"	10"			
D	1	23°29'	00"					
D	3	70°30'	00"	10"				
D	6	140°59'	20"	30"	25	15"	23°29'52.5"	
R	6	0°00'	$\overline{50}$	$\overline{40}$	$\overline{45}$	40"	23°29'56.7"	23°29'54.6"

Date: Nov. 15, 19
Observer: H.A.L.
Inst. Berger No. 7

FIG. 5-3. Angles By Repetition.

an instrument with 10″ verniers, the *A* vernier is read at the first repetition, and both verniers are read carefully at the third repetition, to afford a check, to the nearest minute, on the reading of the sixth repetition. Also, it is convenient to use a bar over a vernier reading when it is less than 0°00′. Thus $0°00'\overline{50}''$ is identical with 359°59′50″.

In the form given, two values are shown in the "Arc" column. The first value is found by subtracting the initial reading from the sixth repetition, ie.,

$$140°59'25'' - 0°00'10'' = 140°59'15''. \qquad (5\text{-}1)$$

The second value is found by subtracting the final reading from the sixth repetition, and in this case it must be observed that the final reading is $0°00'\overline{45}''$, which is $15''$ less than $0°00'00''$, i.e., $0°00'00'' - 15''$. Accordingly, the subtraction becomes

$$\begin{aligned} 140°59'25'' - (0°00'00'' - 15'') &= 140°59'25'' + 15'' \\ &= 140°59'40'' \qquad (5\text{-}2) \end{aligned}$$

Each of the angles (Eqs. 5-1 and 5-2) is now divided by 6 to find the single angles—$23°29'52.5''$, the value measured with the telescope direct, and $23°29'56.7''$, the value measured with the telescope reversed. These values are recorded in the "*D* and *R*" column. The final value of the angle is the mean of the *D* and *R* values.

5-11. Signals A few signals commonly used in transit work are as follows:

Give a Foresight.—When the transitman desires a foresight he signals the head tapeman by holding his arm straight above his head and waving it in a circle about a vertical axis.

Take a Foresight.—When the head tapeman desires to set a transit point, he holds his range pole in a horizontal position above his head and waits until the transitman gives an "all right" signal. The range pole is then brought to a vertical position and lined in for a stake.

Set a Tack.—When a stake has been driven and the head tapeman desires to set a tack, he holds the point of his pole on the stake and waves the top of the pole slightly to the right and left until the transitman answers with an "all right" signal. The point is then lined in on the stake.

Right or Left.—When lining in a range pole the transitman should give definite signals that can be readily interpreted, e.g., he may signal to the right with one arm and to left with the other.

When showing signals, the tapeman should remember that objects are seen by contrast. Hence a taping pin or dark pencil will be seen best in front of a white background, but a white or yellow pencil or the white bands on the range pole will be seen best against a dark background. Of course, illumination is a great aid, and, if convenient, the range pole, pencil, etc., should be held so the sun's rays illuminate the surface toward the instrument.

5-12. Adjustment of the Transit Before proceeding with the description of the adjustment of the transit, the remarks made in connection with the adjustments of the level and the definitions of terms, Art. 3-22, should be reviewed.

The adjustments of the transit should be made in the order given, and usually one set of adjustments will be sufficient. But the relations are interdependent so that if a large correction is necessary in one adjustment it may disturb another. Hence, when a transit is found to be badly out of adjustment, it may be necessary, after having carried through in order one set of adjustments, to begin again and repeat the process, thereby gradually effecting a proper adjustment throughout.

1. ADJUSTMENT OF THE PLATE-BUBBLE TUBES

Relation.—The axis of each plate-bubble tube should lie in a plane perpendicular to the vertical axis.

Test.—This test is the same as that given for the engineer's level in Art. 3-23. The transit is set up and carefully leveled with each bubble tube parallel with a diagonal pair of leveling screws. The plate is then rotated on its vertical axis until each tube is turned end for end over its pair of leveling screws. If the correct relations exist, each bubble will remaind centered; but, if not, the bubbles will be displaced and the amount of the displacement will be double the error of adjustment because of the reversal of conditions which has been made.

Adjustment.—If, upon reversal, a bubble is displaced, say 4 divisions, the adjustment is made by bringing it back 2 divisions by means of the capstan adjusting screw at the end of the bubble tube. The bubble is then centered with the foot screws and the test repeated for verification.

2. ADJUSTMENT OF THE CROSS-WIRE RING

Relation.—The vertical cross-wire should lie in a plane perpendicular to the horizontal axis.

Test.—This test and adjustment are the same as those given under the adjustment of the engineer's level in Art. 3-23, except that in the case of the transit the vertical cross-wire is observed instead of the horizontal cross-wire, as in the case of the level.

The transit is set up and the vertical cross-wire sighted on a definite point in the field of view. The telescope is then rotated slightly

about its horizontal axis. If the correct relation exists, the cross-wire will apparently remain on the point. If not, the point will appear to move off the cross-wire as the telescope is rotated from top to bottom of the field of view (see Fig. 3-18).

Adjustment.—To adjust the cross-wire ring, both pairs of capstan screws which hold the ring in position are loosened slightly so that it may be rotated by means of pressure of the fingers on the screws, or by tapping with a pencil, until the correct position has been obtained.

3. Adjustment of the Line of Sight

Relation.—The line of sight should be perpendicular to the horizontal axis.

Test.—Set the instrument up and level it carefully. Take a backsight on a point as A, Fig. 5-4a, with the telescope in the normal position. Invert the telescope and set a point at D. Reverse the horizontal axis end for end, turning the plate about the vertical axis, and take a second backsight on A with the telescope in the inverted position, Fig. 5-4b. Re-invert the telescope and set a point at E.

The lack of perpendicularity between the line of sight and the horizontal axis is represented by angle α in the illustration; and it is evident that, upon inverting, the line of sight is deflected from the true prolongation of line AB by an angle equal to 2α. Accordingly, after reversal of the horizontal axis and inversion of the telescope the second time, the angle between the two foresights D and E is equal to 4α.

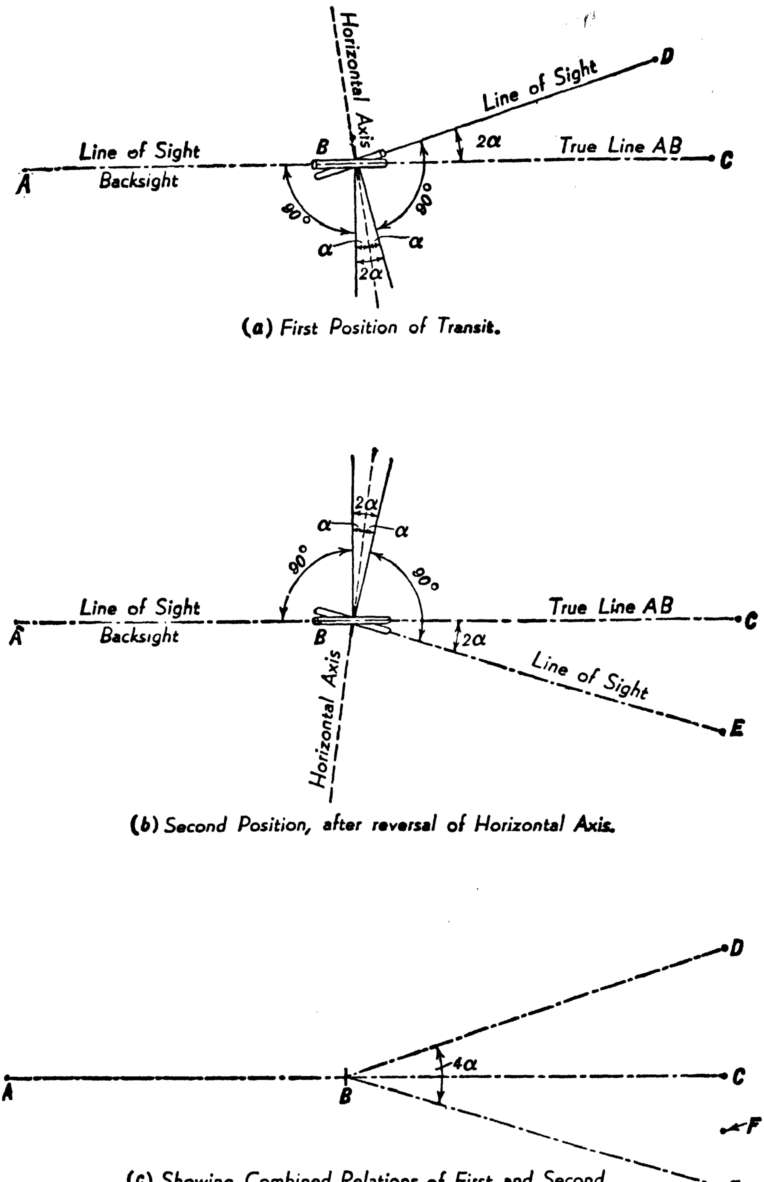
Adjustment.—The adjustment is made by sighting point E, Fig. 5-4c, and then by loosening one capstan screw of the horizontal pair and tightening the other, the vertical cross-wire is fixed on a point F which is set $\frac{1}{4}$ of the distance from E toward D.

After this adjustment the test is repeated and, if the adjustment is perfect, the line of sight will fall on point C both before and after reversal of the horizontal axis.

4. Adjustment of the Standards

Relation.—The horizontal axis should be perpendicular to the vertical axis.

Test.—Set the transit near a building where a definite point can be sighted that requires the telescope to be elevated through a large vertical angle. Level the plate carefully and sight the elevated point.



(c) Showing Combined Relations of First and Second Positions of the Horizontal Axis.

FIG. 5-4. The Effect of Nonperpendicularity Between the Line of Sight and the Horizontal Axis.

Depress the telescope and set point A near the ground. Reverse the horizontal axis end for end by turning the plate about the vertical axis, invert the telescope, and sight the elevated point again. Depress the telescope a second time and, if the adjustment is perfect, the line of sight will fall on point A, previously set. If not, a second point B is set near the ground.

Adjustment.—The adjustment is made by raising or lowering one end of the horizontal axis until, after repeated reversals, the line of sight falls on the same point near the ground. The horizontal axis rests in journals, and provision is made for raising or lowering one of them, usually by first loosening setscrews on top of the standards and then by turning a capstan screw under the journal and between the two legs of the standard.

There is no way of telling exactly how much the standard is to be adjusted, and it becomes a cut-and-try method until the proper adjustment has been made.

5. Adjustment of the Telescope Bubble Tube

Relation.—The axis of the bubble tube attached to the telescope should be parallel to the line of sight.

Test.—The test is the same as that for the relation between the axis of the bubble tube and the line of sight in the engineer's level (Art. 3-23).

Adjustment.—The adjustment is made by sighting the horizontal cross-wire on the correct reading on the rod and then by centering the attached bubble by means of the capstan adjusting screws on the bubble tube.

6. Adjustment of the Vertical Arc

When the plate bubbles and the bubble on the telescope are centered, the vertical arc should read zero. If adjustment is necessary, provision is made for moving the vernier until the index reads zero on the vertical circle.

5-13. Sources of Error. Imperfect Adjustments The principal sources of error in transit work are (1) the imperfect adjustments of the instrument, (2) reading the vernier, (3) sighting, including parallax, (4) setting over a point, (5) soft or unstable ground, and (6) weather conditions, as to wind, heat waves, etc. Each of these

sources will be considered as to the nature of the error, and the manner of reducing or eliminating it.

As regards the use of the transit, the measurement of vertical angles is of minor importance (except in topographic surveying), the principal uses being to measure horizontal angles and to prolong straight lines. Accordingly, it will be desirable to note the effect of each of the adjustments on these principal uses.

It may be noted that the imperfect adjustment of the plate-bubble tubes and of the standards have the same effect, i.e., to incline the horizontal axis with respect to a truly horizontal position. In this condition, when the telescope is rotated about the horizontal axis, the line of sight traces an inclined instead of a vertical line. Hence, in measuring the horizontal angle between two points of some difference in elevation; when the telescope is raised or lowered, the line of sight will follow an inclined, instead of a vertical, line, and the amount of this inclination will be introduced as an error in the magnitude of the horizontal angle. On level ground these imperfect adjustments will have no appreciable effects.

In inverting the telescope when prolonging a straight line, it is evident that, as regards the adjustments under consideration, the same conditions obtain as when measuring horizontal angles. Therefore, the effects of the imperfect adjustments of the plate bubbles and the standards may be stated as follows: (1) they are not important either in measuring horizontal angles or in prolonging lines on level ground, and (2) they do have important effects on both uses of the transit when sighting between points of considerable difference in elevation.

If the line of sight is not perpendicular to the horizontal axis, this condition will not affect the measurement of horizontal angles on level ground, and but slightly when the points sighted are of different elevations. But the full effect is present whenever the telescope is inverted between a backsight and a foresight, as when prolonging a straight line.

It will be remembered that the process given for prolonging a straight line was that of double sights, with the instrument reversed between backsights. It is now evident that this procedure is used to eliminate any effect of the line of sight not being perpendicular to the horizontal axis, for, if the line of sight is deflected to the right when first inverted, it will be deflected to the left when inverted the second time, and the mean of the two sights will be free from error.

It may be added that while the errors resulting from imperfect adjustments are systematic, they are, in general, rendered accidental in the process of taking backsights and foresights; and, further, that most of the instrumental errors will be eliminated by the use of double sights with the instrument reversed between them.

5-14. Other Sources of Error *Reading the Vernier.*—Reading the vernier is the source of an accidental error, for a line of the vernier which appears to match a line of the scale seldom does so exactly, and there is a small remaining error which may have either a plus or a minus sign. The magnitude of the error depends on the fineness to which the vernier reads, on the legibility of the graduations, and other factors. For the average instrument with a vernier reading to 1′ or 30″, the average error of reading the vernier once is probably about 20″. The error is reduced by reading both *A* and *B* verniers, by the use of a reading lens, and by applying the method of repetition.

Sighting.—When sighting a range pole or other object, a small accidental error results because of the difficulty of determining when the cross-wire exactly bisects it. Also, the range pole may not be exactly plumb and the bottom of the pole may not be visible. Parallax also contributes to this source of error. The magnitude of the error depends on the character and visibility of the object sighted and on the length of sight. If an error of 0.1 ft is made in sighting a point 50 ft distant, the angular error is $\tan^{-1} 0.1/50 = 0.002$ or 6′40″, but if the point were 500 ft distant the error would be 40″. Hence, the error is more serious for short sights than long ones. On the other hand, visibility is decreased as the length of sight is increased and so there is always a limit beyond which the error increases.

The error is reduced by avoiding very short sights, by providing good visibility, and by applying the method of repetition. Parallax is prevented by carefully focusing the eyepiece on the cross-wires.

Setting Over a Point.—The error introduced by not plumbing exactly over a point is similar in character and magnitude to that of sighting. It is reduced by careful centering over the point occupied. It should be added that for reasonable lengths of sight and with ordinary care in plumbing over the point, this error will be quite negligible; and many transitmen waste time in being extremely careful in this matter while overlooking other more important sources of error

Soft Ground.—Soft, swampy, or thawing ground becomes a serious source of error if one tripod leg settles more than another and thus twists the graduated plate out of position. The error is prevented either by providing stable supports or by frequently checking the orientation of the instrument.

Weather Conditions.—The use of the transit is affected by such weather conditions as wind, heat waves, extreme heat or cold, etc. The resulting errors are accidental, and their magnitudes depend on the field conditions. They can be reduced in some cases by protecting the instrument from the wind or sun and by limiting the lengths of sights.

5-15. Mistakes The mistakes commonly made in the use of the transit are as follows:

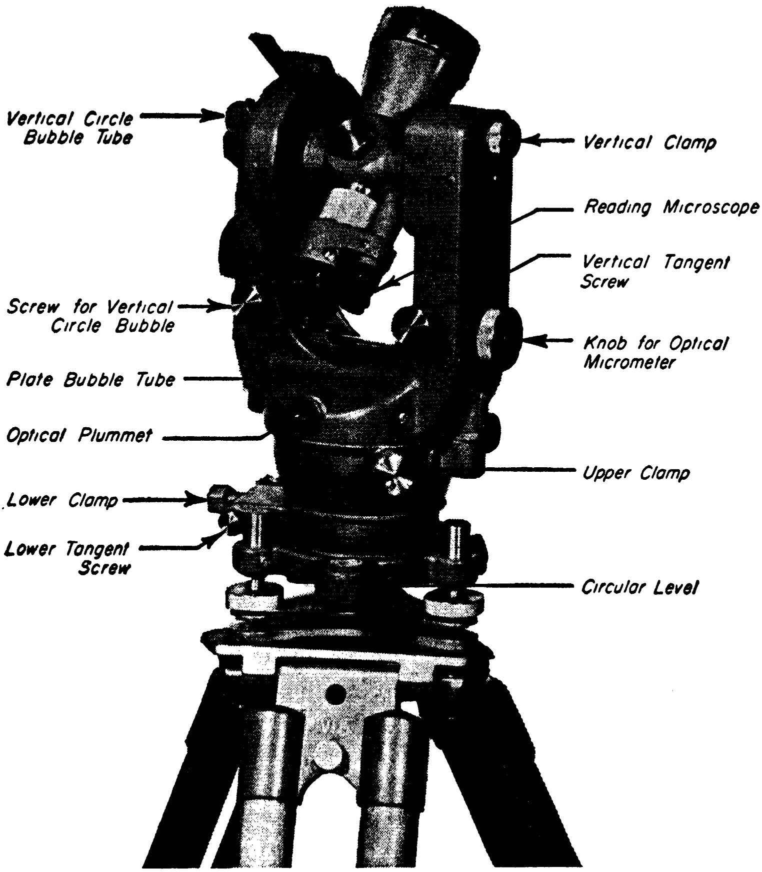
1. *Misreading the Vernier,* which refers to forgetting to add the complete scale reading to the vernier reading; e.g., if the index reads 13°30′ and the vernier reads 06′, the total reading is 13°36′, but is sometimes mistakenly read as 13°06′.
2. *Reading the Wrong Vernier.* This refers to the mistake of reading the right-hand vernier when the left-hand one should be used.
3. *Reading the Wrong Circle.* When the magnitude of an angle is near 180° it is a frequent mistake to read the wrong circle, i.e., the inner instead of the outer circle, or vice versa.
4. *Reading the Circle Incorrectly.* The circle may be read incorrectly, especially for values near the 10° divisions, e.g., 59° is misread as 61°, etc.
5. *Using the Wrong Tangent Screw.* A frequent mistake with beginners is to use the wrong tangent motion; i.e., the upper tangent screw with the lower clamp, or vice versa.
6. *Recording.* This refers to the usual mistakes of recording; such as transposing numbers, etc.

5-16. Optical Transit In recent years a new type of transit has been introduced to the American engineer. This instrument is of European manufacture and differs considerably from the engineer's transit in several significant features of design. Although small and very compact, it furnishes angular values which can be obtained quickly and accurately.

Both vertical and horizontal circle graduations are on glass circles which are completely enclosed and read through a microscope situ-

ated beside the telescope. An optical centering device makes possible the accurate positioning of the instrument over the station mark. This is a marked advantage over the plumb bob on a windy day.

Figure 5-5 shows an *optical transit* which is particularly adapted



Courtesy of Wild Heerbrugg Instruments, Inc.

FIG. 5-5. Wild T-1 Optical Transit.

for surveys of ordinary accuracy. A view of both horizontal and vertical circles is provided in Fig. 5-6. Both circles can be read directly to the nearest minute of arc and by estimation to 0.1′. Another model of the T-1 permits direct readings to the nearest 20″. Horizontal angles can be repeated as with a conventional engineer's transit.

Figure 5-6 shows how the vertical circle is read. This circle reads 90° when the line of sight is horizontal. The micrometer screw, which is located at the bottom of the right-hand support (see Fig. 5-5) of the telescope, is rotated until a graduation in the middle of the vertical circle, marked *V*, is exactly flanked by a pair of movable vertical wires. The number of this graduation mark gives the whole degrees of the reading and the uppermost panel showing the optical micrometer gives the minutes. The horizontal circle, marked *Az*, is read in an identical manner when horizontal angles are measured.

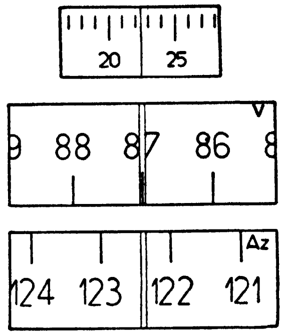


Fig. 5-6. Reading the Wild T-1 Transit.

In Fig. 5-7 is depicted the KE-6e optical transit, which operates in the same manner and performs the same functions as the standard American transit. The horizontal circle is graduated from 0° to 360° in both directions, and the telescope produces an erect image. Both circles can be read to 1′ and by estimation to 6″. The vertical circle is automatically controlled by a pendulum so that the index error (see Art. 5-9) is effectively eliminated.

The chief characteristic that fundamentally contrasts American vernier transits from European optical transits is the optical system of reading horizontal and vertical circles. A representative optical system is shown in Fig. 5-8.

Office Problems

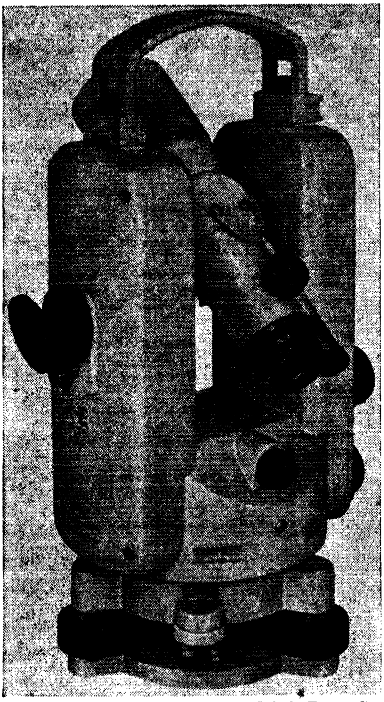
5-1. A transit circle is divided into 20′ divisions, and 60 divisions on the vernier cover 59 divisions on the scale. What is the least value that can be read with this vernier?

5-2. A transit circle is divided into 15′ divisions. Design a vernier that will read to 20″.

5-3. Design a scale and vernier for an architect's rod that will read to 1/32 in.

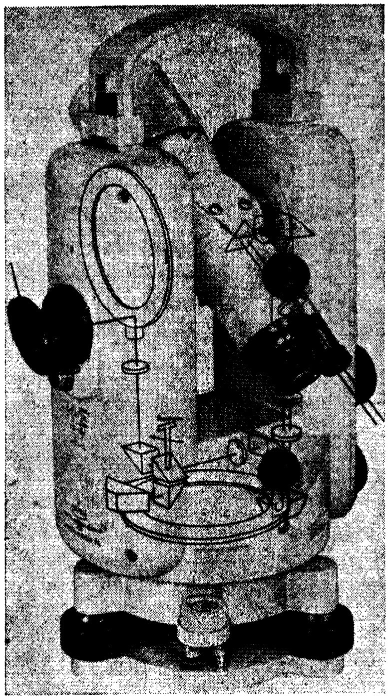
5-4. In sighting a range pole, only the top could be seen, with the result that the line of sight was displaced 0.1 ft. What angular error does this displacement represent if the range pole is 900 ft distant? 500 ft? 100 ft?

5-5. At a certain transit station the plumb bob was ¼ in. (0.02 ft) to one side of the point. What angular error does this displacement represent if the length of sight is 1000 ft? 200 ft? 20 ft?



Courtsey of Keuffel & Esser Co.

FIG. 5-7. KE-6e Optical Transit.



Courtsey of Keuffel & Esser Co.

FIG. 5-8. KE-6e Optical System.

5-6. Find the value of an angle which has been measured by repetition as follows:

Tel.	Reps.	Angle		Verns. A	Verns. B
D	0	0°	00′	00″	00″
D	1	47	10	—	—
D	3	141	30	30	
D	6	283	00	00	30
R	6	0	00	30	00

5-7. A transit is to be used for the following purposes:
(*a*) To plumb the steel columns of a tall building.
(*b*) To prolong the center line of a street over level ground.
(*c*) To run a traverse over hilly ground.
State, for each, what adjustment or adjustments are most important. Explain.

Field Problem 5-1. Reading Angles and Bearings

Procedure.—Set the transit up over a given point and set 6 taping pins in the ground about 200 ft distant and somewhat evenly spaced around the instrument. If each pin is pierced through a sheet of paper it will serve to increase the pin's visibility.

Set the *A* vernier at zero and measure the angle between pins *A* and *B*, having arbitrarily assigned the letters *A*, *B*, *C*, etc. to the series of pins. Again set the vernier at zero and measure the angle between pins *B* and *C*, and so proceed until the six angles have been measured, setting the vernier at zero for each angle.

After the angles have been measured, release the compass needle, sight each pin and read its magnetic bearing.

Record the results as shown in Fig. 5-9.

READING ANGLES AND BEARINGS ABOUT A POINT

Obj.	Inst. at Sta. O Angle	Mag. Bear.	Angle from Bear.	Diff.	
A		N35°15'E			
	38°12'		38°00'	−12'	
B		N73°15'E			
	63°38'		63°45'	+07'	
C		S43°00'E			
	61°14'		61°30'	+16'	
D		S18°30'E			
	58°44'		58°45'	+01'	
E		S77°15'W			
	75°23'		75°15'	−08'	
F		N27°30'W			
	62°46'		62°45'	−01'	
A	359°57'				

Locker No. 14

J. G. Griffith ⊼
W. B. Dickman ⊼
July 8, 19
32

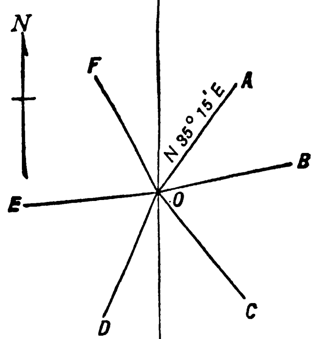


FIG. 5-9. Field Notes for Angles and Bearings.

The sum of the angles about the station should equal 360°, and a permissible error of 03′ is allowed. If the first measurement of the angles yields an error greater than 03′, each angle should be checked by measuring it again, until the sum comes within the permissible error.

Calculate the angles from the bearings and find the difference between each angle thus found and its value as measured by the transit. The maximum permissible difference is 30′.

Field Problem 5-2. Prolongation of a Straight Line

Procedure.—Set the transit over a point *A* and establish point *E* about 1200 ft distant. Without moving the transit and on the same line set point B about 300 ft distant. Move the transit to point *B*, set up and prolong line *AB* by the method of double sights, thus set-

ting point C about 300 ft from B. In like manner, set point D about 300 ft from C, also set up at D and set point F opposite point E which was first established.

Point E should not be visible as a foresight when the intermediate points are being set.

Use a sketch and describe the work according to good notebook practice.

A check is furnished at stakes D, E, and F, in that the difference between the double foresights should be the same, provided the lengths of the sights are the same.

The final check is the offset distance EF between the initial and final points. For the given conditions, this distance should not exceed 0.2 ft.

Field Problem 5-3. Measurement of Angles by Repetition

Procedure.—Set the instrument up carefully and establish good foresights on all points to be observed. Range poles plumbed with a plumb line or rod level and guyed with cords serve well as foresights. Follow the procedure indicated in Art. 5-10, and measure each angle about the station. Record the results as indicated in Fig. 5-3.

The checks provided are (1) the reading of the vernier on the first and third measure of each angle; and (2) the sum of the angles about the station should equal 360°. The permissible error of closure may be taken as 5″ times the number of angles measured.

Field Problem 5-4. Adjustment of a Transit

Procedure.—Follow the procedure indicated in Art. 5-12. Use great care in all operations and be sure that, when through, all adjusting screws have been brought to a firm, but not tight, bearing. After all adjustments have been made, the tests should be repeated for verification. Make a brief but complete record in the notebook of each test and adjustment.

CHAPTER 6

TRANSIT SURVEYS

6-1. Remarks A network of survey points whose horizontal positions have been accurately determined is called *horizontal control.* Horizontal control may be established either by *traverse* or *triangulation* or a combination of the two methods.

Any traverse consists basically of a series of lines whose lengths and directions have been measured, connecting points whose positions are to be determined.

Traverses are of two kinds. If the series of lines forms a closed area, it is said to be a *closed* traverse; if not, it is called an *open* traverse. A closed traverse provides valuable checks on the field work which an open traverse does not afford, and accordingly it should be employed whenever conditions are favorable. Each kind is illustrated in Figs. 4-1 and 4-2.

Triangulation is a method for extending horizontal control which requires few length measurements and numerous angle measurements. The survey stations are points on the ground which define the vertices of triangles forming parts of quadrilaterals or chains of triangles. The horizontal angles at each station are measured, and the lengths of the triangle sides are determined through the application of trigonometry by means of successive computations through the chain of triangles from a side of known length. This side, the *baseline,* is directly measured in the field.

For surveys of limited extent, the traverse method is most satisfactory. The triangulation method is generally preferred in rough or hilly terrain where numerous promontories are available for station sites which are easily intervisible and where it would be difficult or impossible to conduct taping measurements. The method is used for practically every purpose and with every degree of precision that

apply to traversing, so that the choice between the two methods depends upon field conditions only and not upon the consideration of the accuracy of results.

It is not within the scope of this book to treat triangulation in detail, but it is believed that the theory and practice of the method for small areas are so simple that a brief account of the procedure will be sufficient. For more extensive surveys, more thorough discussions of the subject will, of course, be necessary.

The survey points having been located either by traversing or by triangulation, the details of a survey may be referred to the traverse or triangulation stations by any of the various methods of locating points and lines. Such details may be for plotting a map, for calculating areas, or for any purpose the survey is to serve.

TRAVERSE

6-2. Routes and Stations The transit and tape traverse is perhaps the most common and most widely used field operation of surveying. Whether it be utilized for providing basic horizontal control for extensive surveys or be executed as part of a land, route, or construction survey, certain guiding principles should be observed with respect to the selection of routes and the marking of stations.

All traverse lines should either form closed loops, or begin and end at points whose positions have been fixed by other surveys of higher quality. Ordinarily, the route of the traverse will be dictated by the requirements of the situation as in the case of making a boundary survey of a tract of land. When the traverse is to furnish horizontal control for a mapping project covering an extensive area, the route will follow highways and railroads in order to facilitate transportation and the execution of accurate measurements. In general the route should be carefully planned so that the data obtained by the traverse will satisfactorily and economically serve the purposes for which the traverse was executed.

Another matter of great importance is that the traverse stations be durably monumented if they are to perpetuate the results of the survey. Hence, the marker should be substantial, adequately referenced to nearby objects, and carefully described.

6-3. Distance Measurement The field work of traverse is generally divided into two basic operations, taping and angle measurement. There are, however, indications that electronic procedures

(see Art. 2-17) for measuring lengths on traverse will gain increased acceptance in the future.

The procedures for measuring distance by taping have been discussed in Chapter 2. The importance of good taping cannot be overemphasized. It is the major factor affecting the accuracy of traverse and it is the most significant element influencing the productivity and hence the cost of the traverse survey. For transit traverse intending to serve the requirements of most engineering projects the quality of the distance measurements should be such that the error of any course shall not exceed one part in 5000 parts of the length of the course.

6-4. Angle Measurement The angle measured at a traverse station serves basically to express the difference in the directions of the two lines at that point. Although any angle between the lines will satisfy this purpose, several kinds of angles have been widely used. Accordingly, traverses are sometimes designated by the type of angle that was measured.

1. Deflection Angle Traverse

The deflection angle was defined in Art. 4-3. This angle can be turned either to the right or left of the preceding line extended. When the deflection angle is small, the transitman is likely to make a mistake in denoting the direction in which it was measured. For this reason the deflection angle is no longer extensively used. For a closed traverse the algebraic sum of the deflection angles should equal 360°.

2. Angle-to-Right Traverse

The angle-to-right has very largely supplanted the deflection angle, especially for open traverses. For a closed figure the sum of the angles-to-right should equal $(n + 2)180°$, where n is the number of sides in the polygon forming the traverse.

It should be noticed that for a closed figure, if the angle-to-right traverse proceeds in the counterclockwise direction, interior angles will be measured. Figure 6-1 indicates that the transit is placed initially at station A, where a backsight is taken to E. The angle-to-right is then measured to station B, the forward bearing is observed, and distance AB is measured. The other stations are then occupied in order around the field and the record is kept as shown in Fig. 6-1.

The sum of the interior angles of a polygon equals $(n-2)180°$, n being the number of sides, and thus a check is provided on the accuracy of the measurements. In the given example the sum of the angles shows an error of 03′.

ANGLE TO RIGHT TRAVERSE
South of Engineering Hall

Sta.	Dist.	Angle to Right	Adjusted Angle	Mag. Bear.	Calc. Bear.
A		83°08′	83°08′		
	1122.1			S 87°00′E	S 87°00′E
B		80°42′	80°42′		
	738.7			N 6°30′W	N 6°18′W
C		65°23′	65°22′		
	558.7			S 58°00′W	S 59°04′W
D		232°07′	232°06′		
	495.3			N 69°15′W	N 68°50′W
E		78°43′	78°42′		
	575.4			S 9°30′W	S 9°52′W
A					
		540°03′	540°00′		

Locker 20
May 12, 19

Hedrick ⊼
Hermeson Tape
Jacobs "

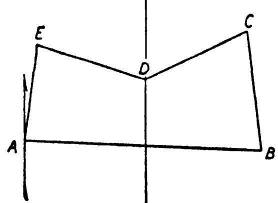


FIG. 6-1. Field Notes for Angle-to-Right Traverse.

Before computing the calculated bearings, the 03′ error in the angles should be distributed among them. Theoretically, the error should be divided equally so that the sum of the adjusted angles should equal 540°, but this adjustment would yield angles computed to seconds of arc, which would be unnecessary accuracy for angles measured to minutes only. Hence, it will be a satisfactory adjustment if the 03′ error is adjusted by subtracting 01′ from each of three angles, say C, D, and E. In this manner the adjusted angles have been obtained. The calculated bearings are then found by beginning with the first observed bearing, $AB = \text{S } 87°00' \text{ E}$, and computing the others, using the adjusted angles. The observed magnetic bearings serve only as an approximate check of the calculated bearings.

3. AZIMUTH TRAVERSE

In order to execute an azimuth traverse, assuming that the stations have been set and that the transit is in position at one of them, a reference meridian must first be chosen. This may be either a true, magnetic, or an assumed meridian, depending on field conditions.

The use of a true meridian requires astronomical observations which are described in Chapter 15. If a definite and conspicuous object is present in the vicinity, it may be sighted from the initial station, and this direction used as an assumed meridian. Otherwise, the magnetic meridian may be used, and in this article it will be supposed that this method is employed.

The instrument is set up at the first station, the A vernier is set at 0°00′, the needle released, and by use of the lower motion the line of sight is fixed in the magnetic meridian and the lower motion is clamped. Next, the upper motion is released and the last preceding station of the traverse, i.e., station E, Fig. 6-2, is sighted and the

AZIMUTH TRAVERSE

Sta.	to Sta.	Dist.	Observed Azimuth	Adjusted Azimuth	Mag. Bear.	Calc. Bear
A	E		92°37'			
A	B	575.4	9°30'	9°30'	N 9°30'E	N 9°30'E
B	A		189°30'			
B	C	495.3	110°48'	110°48'	S 69°15'E	S 69°12'E
C	B		290°48'			
C	D	558.7	58°41'	58°42'	N58°30'E	N58°42'E
D	C		238°41'			
D	E	738.7	173°17'	173°19'	S 7°00'E	S 6°41'E
E	D		353°17'			
E	A	1122.1	272°34'	272°37'	N87°00'E	N87°23'W
	92°	37'+180°=	272°37'			
			03'	Error		

Locker 19
May 14, 19

Moriarty ⊼
Olsen Tape
Douglas "

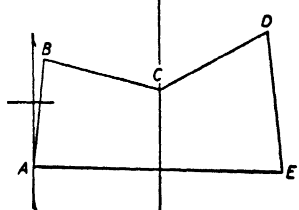


Fig. 6-2. Field Notes for Azimuth Traverse.

azimuth, 92°37′, of this line is observed and recorded. Without disturbing the lower motion the next station, B, on the traverse is sighted and the azimuth, 9°30′, and bearing, N 9°30′ E, are observed. Distance AB is measured and the instrument is taken to station B.

After leaving station A, the transit is oriented at each succeeding station, not by use of the needle, which would be too inaccurate, but by a backsight on the last preceding station.

With the transit in position at station B, the A vernier is set to read the back azimuth of AB, i.e., $9°30' + 180° = 189°30'$, and then, using the lower motion, a backsight on A is taken. The upper motion

is then released, station C is sighted, and the azimuth, 110°48′, and bearing S 69°15′ E are observed.

At station C, the backsight on B is taken with the vernier reading $110°48' + 180° = 290°48'$, and then the foresight is taken. This procedure is followed in order around the field including the last station E. Here the azimuth of EA is found to be 272°34′, whereas the azimuth of AE as observed with the transit at A was 92°37′. Obviously, these azimuths should differ by exactly 180°, but a discrepancy of 03′ is found, which in this case, represents the angular error of closure in the traverse from all sources, just as the sum of the interior angles indicated the same error in the previous example.

It should be noted that if the back azimuth of EA had not been observed at the initial station A, the direct field check would not have been obtained at station E, and another setup at station A would have been required to obtain this necessary check.

As in the case of the angle-to-right traverse, before the calculated bearings are computed, the 03′ error is adjusted. For this traverse this is done by increasing the azimuth of CD by 01′, DE by 02′, and EA by 03′. These adjusted azimuths when changed into bearings, constitute the calculated bearings.

Another method of orienting the instrument at successive stations is sometimes used, by which, when making the backsight, instead of setting the vernier to read the azimuth of the preceding line, the vernier is left clamped as set at the preceding station, and the telescope is inverted to make the backsight. Then the telescope is returned to the normal position, the upper motion released, and the foresight taken on the next station. The bearings are observed, of course, as usual. There is an important objection to this method in that a mistake or an error in reading the vernier at any given station will not be made evident in subsequent calculated bearings or in the final check on the angular error of closure. Accordingly this method is not recommended.

Before closing this description of traverse angles, it is appropriate to mention that deflection angles and angles-to-right are frequently doubled. By doing this three important advantages are gained: first, the accuracy of the measurement of each angle is increased; second, a check is applied to detect any mistake in operating or reading the instrument; and third, most of the errors arising from the imperfect adjustment of the instrument are eliminated.

By this method the angle to be observed is measured once in the

usual manner. Then leaving the upper clamp set, the lower motion is released, the telescope is inverted, and a second backsight is taken. Then the upper motion is released and a second measurement of the angle is made. Now the circle reading gives the double value of the angle. The record is kept as shown in Fig. 6-3.

Angle to Right Record

Station Occupied	*From To*	*Circle* ° ′	*Angle* ° ′
	37	0 00.0	
38		17 28.8	
	39	34 57.8	17 28.9
	38	0 00.0	
39		121 18.1	
	40	242 36.2	121 18.1
	39	0 00.0	
40		183 14.1	
	41	6 28.3	183 14.15
	40	0 00.0	
41		201 15.7	
	42	42 31.6	201 15.8
	41	0 00.0	
42		195 27.9	
	RPS–16	30 55.6	195 27.8

Wild T-1 No. 3785
Windy
Temp. 45°F

Oct. 27, 19
R.P. Smith – Transit
W.J. Rockford
M.T. Bruns } *Rodmen*

Traverse along County Highway H from Elkhorn to Tibbits.

RPS–16: RR spike in bituminous pavement, west side of road, opposite entranceway to farm of J.P. Hollingsworth; 0.8 mile South of Tibbits.

FIG. 6-3. Field Notes for T-1 Transit.

The maximum allowable accumulated angular error in a traverse is usually expressed as a coefficient times the square root of the number of stations occupied. A very common specification for good quality work is $\pm 30'' \sqrt{N}$.

6-5. Relation Between Angles and Distances In all surveying measurements involving angles and distances, it is desirable that a consistent relation be maintained between them if the best results are to be obtained. This relationship is shown in Fig. 6-4, where it is supposed that point B is to be located with respect to point A, and that this requires the measurement of a distance D, and an angle α. Each measurement is subject to error, and let these be represented by E_d and E_α respectively. The linear displacement of B with respect to A due to the angular error E_α, is E_a, and is given by the product $D \times \tan E_\alpha$. If the two sources of error are consistent, then E_d will be equal to E_a.

An error in distance is expressed as a ratio of the error to the

distance measured and is reduced to a fraction whose numerator is unity (see Art. 2-15). Hence, if a distance has been measured with a precision such that the error is 1/1000, and if an angle is to be measured with a corresponding precision, the permissible error will

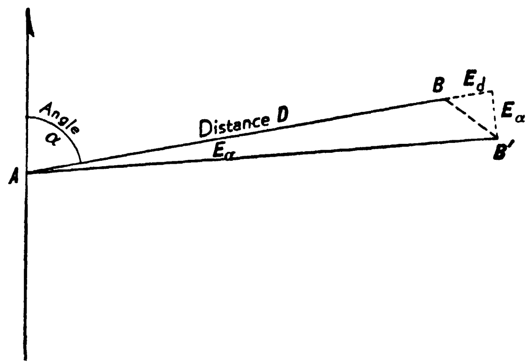


FIG. 6-4. Relation Between Angles and Distances.

be such that the tangent of the angle $E\alpha$ will also be 1/000 or 0.001. Since the sines or tangents of small angles may be considered to vary directly with the angles, and since the tan of 01′ is 0.0003 (nearly), then it is evident that, if the accuracy of a measured distance is 1/1000, the corresponding accuracy of a related angle is 0.001/0.0003 = 03′. Similarly, if a distance is measured with an accuracy of 1/10,000, related angles should be measured with an accuracy of 0.0001/0.0003 = 20″.

In general, it can be said that there will be consistency between linear and angular measurements if the relative error in distance equals the angular error in radians.

6-6. Field Notes In keeping the notes of a transit survey, a sketch or sketches are nearly always used, especially to illustrate the manner of beginning or ending the work.

Distances, azimuths, and bearings always relate to lines, and such lines are designated by the sequence of letters in the station column. Accordingly, the data which pertain to a given line, as *BC*, should be recorded on that line in the notebook, indicated by the station letters *B* to *C*. The instrument station is always given as the first letter in the sequence.

For every transit survey it is important that the initial station be carefully described and referenced (see Fig. 6-7) and the manner of orienting the transit be explained, so that anyone can at a later time, if desirable, occupy the same station and retrace or extend the original survey. This information is often necessary also to enable a draftsman to plot the notes.

All field checks obtained should be plainly indicated.

6-7. Checks The checks which are commonly applied to the measurement of angles include: (1) the comparison of the observed and the calculated magnetic bearings, (2) the sum of the interior or exterior angles of a closed traverse, (3) the final forward azimuth of a closed traverse should differ from the back azimuth of the same course by exactly 180°, and (4) each angle is doubled, the vernier being read each time. On extended surveys, astronomical observations are used to check the transit work. Thus at intervals of from two to ten miles, depending upon the number of setups and the desired accuracy, an observation is made on Polaris to determine the true azimuth of a line. This observation checks all angles measured since the previous check. However, proper allowance must be made for convergency of the true meridians.

Astronomic observations for azimuth provide expressions of direction which are referred to lines or meridians which converge when extended to the poles. Therefore, the surveyor should realize that, when executing a traverse for several miles either eastwardly or westwardly from an initial line of known astronomic azimuth, there will be introduced an effect that may cause the azimuth of the last line as deduced from the measured horizontal angles to be in significant disagreement with its observed astronomic azimuth even though no errors are present in the traverse angles.

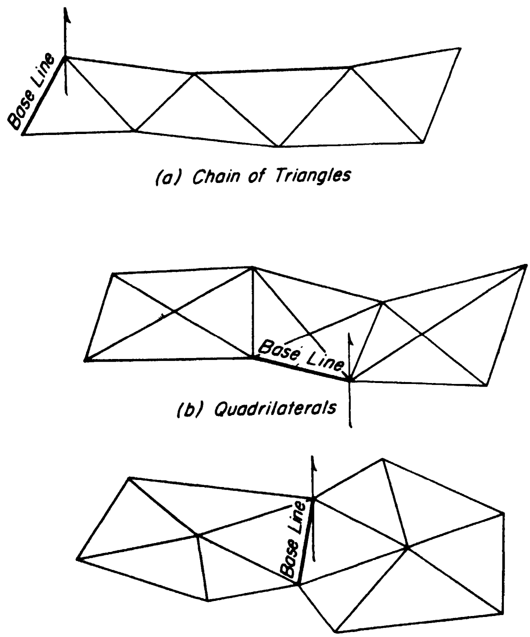
A rigorous method of calculating convergency can be utilized. Here, however, it will suffice to say that in the middle latitudes of the United States, the magnitude of the convergency correction is approximately 0.7′ per mile of east-west distance. In extending westwardly a traverse that has a total departure of 10 miles, it can be expected, therefore, that the astronomic azimuth of the terminal line should be approximately 0°07′ less than its value determined by traverse, even though all field angles are without error.

For a line extended to the east, the correction will be of the same magnitude with the opposite sign.

The measurement of distance is checked by the calculated error of closure of closed traverses, but such measurements can be verified for open traverses by duplicate measurements only. Ordinarily these need not be as precise as the original; for example, stadia measurements may be checked by pacing, or taped distances may be checked by stadia, etc. No dependence can be placed on any measurement, however, until it has been checked.

TRIANGULATION

6-8. Triangulation The three systems of triangulation most commonly used are illustrated in Fig. 6-5. Here it is seen that if one



(a) Chain of Triangles

(b) Quadrilaterals

(c) Central-Point Figures

FIG. 6-5. Triangulation Systems.

side in each system, called the *baseline, is* measured together with the angles, then all other sides can be computed. Also, if a reference meridian is chosen, or a true meridian is observed, the bearings of all lines and coordinates of all points can be found.

The simple chain of triangles is suitable for surveys of ordinary

precision and for a comparatively narrow strip, such as a river valley. It is not as accurate as the other systems and does not provide as many checks. Baselines are introduced into the system at intervals to prevent the accumulation of errors; a check is provided also, since the length of any baseline, as computed from the one next adjacent, should equal its measured length.

The system of quadrilaterals is excellent and is adaptable to many field conditions. It permits the lengths of all sides to be computed by two independent sets of data, thus affording a check on all computations as the work proceeds.

The central point figures are suitable for wide areas and they also yield precise results and convenient checks on the work.

The necessary steps in the application of the method are as follows: (a) location of stations and erection of signals; (b) measurement of a baseline; (c) measurement of angles; (d) determination of a reference meridian; (e) computation of lengths of sides and coordinates of points.

6-9. Stations and Signals The first step in the field is that of locating the system of stations and signals such that they will form well-shaped triangles and that adjacent stations are intervisible. It is desirable that no angles in the system shall be less than 30° or more than 150°.

Since the permanence of the survey depends on the objects used to mark the stations, these should be of durable materials and well referenced. Iron pipes driven into the ground or encased in concrete are commonly used. On rock ledges or boulders, an iron or copper plug cemented into a hole drilled in the rock makes a good marker. The signals may be an ordinary range pole guyed in position, or a pointed mast of larger proportions may be provided.

To establish intervisibility between stations, it is sometimes necessary to raise the instrument above the ground and support it on some kind of platform. Such preparations are made, of course, before observations begin. An example of a rather elaborate observing platform and elevated signal mast is illustrated in Fig. 6-6.

6-10. Baseline Measurement A baseline is chosen usually along a graded highway or a railway; but, if necessary, very good results can be secured over fairly rough and sloping ground.

If the baseline is located along a paved highway or a railway, no

special preparations are necessary, but if the line must be measured over rough ground, some preparations must be made. Provision for marking the tape lengths is made by driving substantial posts (2 × 4 in.) at each tape length, on which are placed cardboard or

Courtesy of California Dept. of Public Works

FIG. 6-6. Triangulation Tower.

zinc strips to receive the end marks. Between the 100-ft posts, supports are placed at the desired intervals, i.e., either 50 ft or 25 ft apart. These supports consist of board slats, perhaps 1 × 2 in., driven accurately on line. Into each board a small nail is driven, on line with the tops of the two adjacent 100-ft end posts. The tape is then

stretched over the nails and so supported at regular intervals on a uniform grade between posts.

Additional equipment includes two thermometers, a good spring balance for applying tension and a finely divided scale for reading small distances.

The party consists of five men, namely, two tapemen, one marker, one observer, and a recorder.

The procedure when measuring over uneven ground is as follows: the tape is stretched by the two tapemen, each of which carries a pointed staff to be placed in the ground and around which the tape leather is looped. By this means a steady pull can be exerted. The head tapeman gives the proper tension by use of the spring balance, and the rear tapeman insures that his end of the tape is in position when each measurement is made. The observer notes carefully when the rear end of the tape is in exact coincidence with the mark and at that time gives an "all-right" signal to the marker who marks the position of the head end of the tape by making a sharp pencil mark on the paper strip.

As the work proceeds the head end of the tape may fall short of, or beyond, the post in place. In this case, a short distance is measured on the last marked strip, thus to set the tape forward or backward as may be required. Such a measurement is called a *set-forward* or a *set-back,* as the case may be, and is recorded in the notes to be added to, or subtracted from, the total measured length of the line.

In general, it is desirable that at least two baselines be used in any project, one near each end, so as to provide a check on the work.

6-11. Errors in Baseline Measurement The effects of the various sources of error in tape measurements have been discussed previously (see Art. 2-11), but a few remarks will be appropriate here as regards these errors in baseline measurements.

Length of Tape.—A light tape (not over $1\frac{1}{2}$ lb) should be used, and it should be standardized by the U.S. Bureau of Standards, Washington, D.C. If the tension and unsupported interval to be used in the field are specified, the standard comparison will be made under the same conditions. This eliminates the necessity of computing sag and tension corrections to the field measurements.

Temperature.—In base measurements the effect of temperature changes is important. For this reason such measurements should, if

possible, be made under conditions of constant temperature, as on a cloudy day, or at night. A thermometer should be read each time the tape is stretched and should be held at the same height as the tape above the ground.

Grade.—The effect of grade is determined by running a line of profile levels over the tops of the 100-ft posts described above. From these data the proper correction can be computed and applied as explained in Art. 2-9.

Sag.—The effect of sag will need no correction if the same unsupported interval and tension are employed as those used by the Bureau of Standards. Otherwise the proper correction may be computed.

Marking.—The error due to marking the tape lengths can be rendered negligible if a sharp pencil is used.

Wind.—A strong wind transverse to the line should be avoided, because its effect may easily amount to as much as that of sag, and it is difficult to make the proper corrections.

6-12. Angle Measurement The various sources of error in measuring angles with a transit have been discussed in Art. 5-13. If care is taken with respect to these sources it will not be difficult to secure the desired precision. Noticeable improvement in the results will be secured if the instrument is protected from a strong wind or the sun's rays. The method of repetition is employed as explained in Art. 5-10, using six repetitions, direct and reversed. The use of an optical transit is recommended.

All angles at each station should be measured and their values adjusted to equal 360°. Also the angles in each separate triangle should be adjusted to equal 180°.

6-13. Specifications The general specifications for the character of work contemplated in this chapter are that the total error in the length of the baseline shall not exceed 1 part in 50,000; and that the discrepancy between the measured length of a line and its length as compared from the next adjacent baseline shall not exceed 1 part in 5000. The maximum angular error of closure in any triangle should not exceed 30″. The lengths of sides of the triangles may vary from a few hundred feet to perhaps a mile or two in length, and the base lines from 1000 ft to 2500 ft in length.

The desired precision in the measurement of the angles will be secured by the methods indicated in Art. 6-12.

The desired precision in the base measurement will be secured by conforming to the following detailed specifications:

1. The tape shall be of good quality and standardized by the U.S. Bureau of Standards.
2. The mean average temperature of the tape shall be determined within 0.5°F.
3. The effect of slope shall be corrected by determining the elevation of the tops of the posts within 0.05 ft.
4. The tension on the tape shall not vary more than 0.5 lb from the standard.
5. The marking shall be exact within 0.002 ft.

MISCELLANEOUS OPERATIONS

6-14. Random Line When it is desired to connect with a straight line two distant points that are not intervisible, it is frequently accomplished by projecting from one point a straight line that is estimated to fall nearly upon the other point. When the line thus projected has been run out, its position relative to the distant point is measured, usually by a swing offset, and from these data the direction of a true line between the two points is calculated and established on the ground. The projected or trial line is called a *random* line. This procedure is much used in land surveying to establish intermediate points on a straight line between two corners.

6-15. Referencing Points The purposes of many surveys are served within a short period of time and the permanence of field stations is not important. This is never true of land surveys, however, and on many construction surveys it is required that controlling points and benchmarks be protected and undisturbed during the construction period. The young engineer is likely not to appreciate either the importance of this matter, or the difficulty of establishing a point permanently. Accordingly, the matter of referencing points should receive his careful attention.

Three types of points to be referenced are shown in Fig. 6-7.

Fig. 6-7a illustrates an alignment point (*P.C.*) of a route project which is to be referenced during the construction period. This may be accomplished in various ways, but a good method is that which

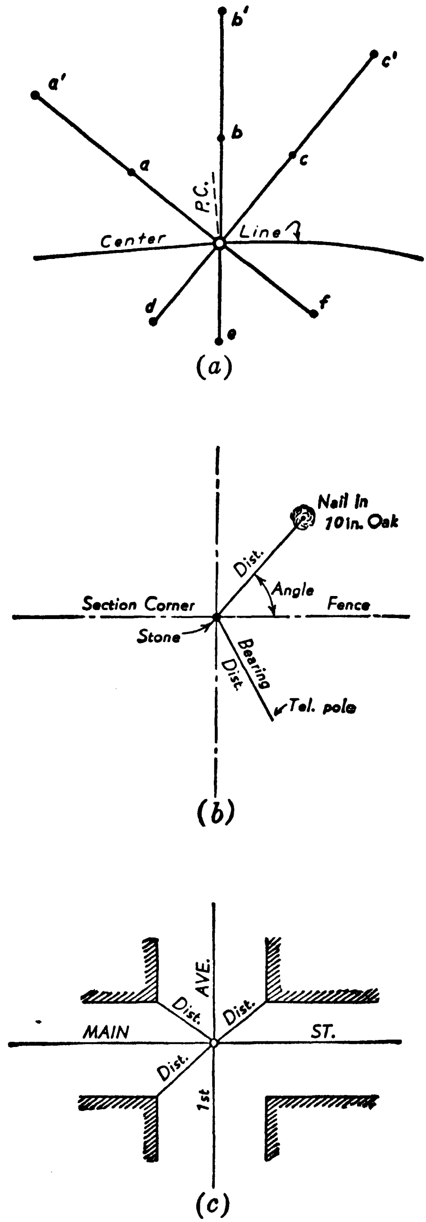


FIG. 6-7. Reference Points.

uses straight lines only. Points *a, b, c, d, e,* and *f* are solid stakes with center tacks, set outside the limits of construction operations, fixing lines *af, be,* and *cd,* passing through the *P.C.* Two of these lines would fix the position of the *P.C.* but the third line provides a check. If it is not convenient to establish points *d, e,* and *f* on the opposite side of the center line from *a, b,* and *c,* then points *a′, b′,* and *c′,* may be used. After construction is completed, the *P.C.* is relocated by projecting the lines to an intersection.

Fig. 6-7b illustrates a section corner referenced by a tree and telephone pole. Various combinations of distances and angles may be used, but the principal considerations are the permanence of the objects used and that sufficient measurements be taken to provide one or more checks on the relocation of the corner.

Fig. 6-7c illustrates a street intersection. The masonry of adjacent buildings affords the best reference objects, although concrete sidewalks, pavement curbs, etc., are sometimes used. Because of the frequent changes in city structures it is most difficult to retain the positions of city street lines permanently, and hence the matter of referencing points becomes of great importance in city surveys.

6-16. Locating Points and Objects Where many points and objects are to be located, they are usually referred to a transit line run conveniently adjacent, stakes being set at each 100-ft station to serve as reference points. Sometimes a sidewalk or pavement having the 100-ft stations marked with keel will serve as a base line.

The common methods of locating points and lines are illustrated in Fig. 6-8, and may be classified as follows:

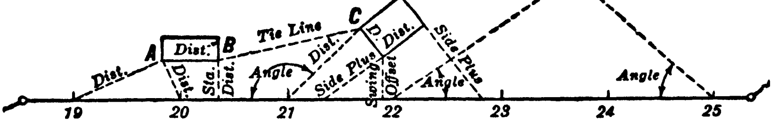


FIG. 6-8. Locating Points and Objects.

1. *Distance from Two Points.* Point *A*, the corner of a building, is located by the two distances from stations 19 and 20.

2. *Rectangular Coordinates.* Point *B* is located by rectangular coordinates, and is the station plus to the end of the building and the perpendicular distance thereto, respectively.

3. *Polar Coordinates.* Point C is located by polar coordinates, and is the angle and distance measured from station 21.

4. *Two Angles.* Point D is located by the two angles measured at stations 22 and 25.

Various combinations of these methods may be used, and the application of other geometrical principles will be evident as the field work proceeds. Thus, as shown, a building whose sides are not perpendicular or parallel with the transit line may be located by a swing offset and by noting the station numbers where the sides prolonged intersect the transit line. Important points should be located by more than one method, and tie lines should be used to check both the field measurements and the plotting.

6-17. Passing Obstacles A traverse line is frequently obstructed, either to vision or measurement, and the line must be carried by the obstacle in some way. The most common methods are illustrated in Fig. 6-9, *a*, *b*, and *c*. In Fig. 6-9a line AC is carried past the pond by means of equilateral triangle CED; also, this may be accompanied by the similar triangles FVG and fVg. Angle δ is the same at F, f, g, and G. Then $fg/FG = fV/FV$.

In Fig. 6-9b, the obstacle is passed by means of right angles.

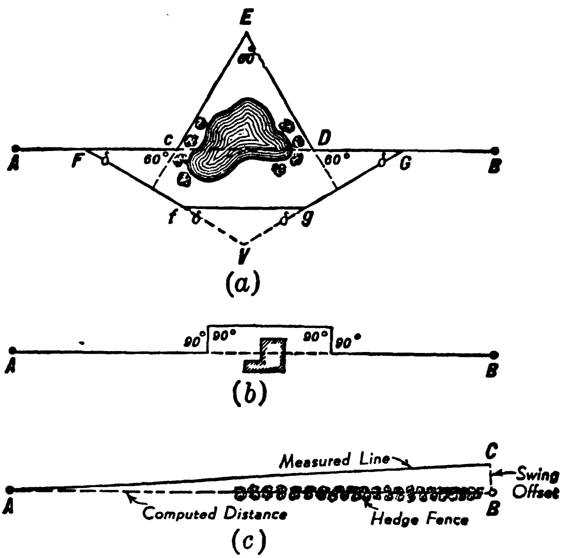


FIG. 6-9. Passing Obstacles.

In Fig. 6-9c, a hedge fence on line is passed by means of a right-angle triangle ACB. The direction of AC is chosen so as to avoid the obstacle and is measured. From point B a swing offset is taken to line AC and measured. From these data the direction and length of AB can be computed.

Office Problems

6-1. Referring to Fig. 6-9c, the measured distance AC is 1325.6 ft. The swing offset BC is 24.3 ft. What is the length AB (a) by the approximate method of Art. 2-9? (b) by the exact method?

6-2. The conditions under which a baseline was measured were as follows: the tape was 100.002 ft long, under a tension of 20 lb and supported throughout its length, at 62°F; weight was 1.3 lb; coefficient of expansion, 0.0000065 per degree F; the mean measured length of the line was 1427.324 ft; the uniform grade was 2.3%; the tape was supported at 50-ft intervals under a tension of 20 lb; the average temperature was 84.3°F. What is the corrected length of the line?

6-3. In a system of triangulation, triangle ABC was measured as follows: $A = 57°44'40''$, $B = 44°27'40''$, $C = 77°48'10''$. Side $AC =$ 4022.28 ft, azimuth 96°10'40''. The coordinates of station A, $Y =$ 6000.00, $X =$ 6000.00. Station A is north of side BC.

Adjust the angles in the triangle and find the lengths of sides AB and BC.

6-4. For the triangle of Prob. 6-3, find the coordinates of points B and C from station A; find also the coordinates of point C from station B.

Field Problem 6-1. Height of Object

Object. To determine the height of an inaccessible object above a benchmark.

Equipment. The equipment includes a transit, plumb bob, tape, stakes, tacks, and level rod.

Procedure. Choose some definite high point on a building and a benchmark. Set up the transit where both objects are visible and distant perhaps 200 to 300 ft. Let the vertical projection of the high point be A, the point itself A', and let the transit station be B. Set up the instrument over B, level it carefully, and measure the vertical angle to A'. Lay off a right angle ABC to a point C at a convenient carefully measured distance, say 100 ft. Determine the height of the instrument by means of a level reading on the benchmark.

Move the transit to C, measure the vertical angle to A', measure the horizontal angle BCA, and take a level reading on the benchmark.

These observations provide sufficient data for two independent

determinations of the height of the elevated point above the benchmark.

Checks. The two computed elevations of the high point should agree within 0.3 ft.

Field Problem 6-2. Locating Points and Objects

Procedure. Run a transit traverse near a number of buildings and other definite objects such as water hydrants, light poles, trees, etc. Drive numbered stakes at 50- or 100-ft intervals along the traverse. Locate the various objects by one or another of the methods indicated in Art. 6-16. Be careful that enough measurements are taken so that several checks can be applied when the data are plotted on a map. Use a sketch and be careful that the recorded data are complete and legible.

CHAPTER 7

COMPUTATIONS

7-1. Remarks Computations are required in connection with most surveying measurements to reduce them to convenient form for plotting, or to calculate distances, areas, and volumes, or for other purposes.

Three essential characteristics of engineering computations are that they should be in permanent form, arranged in an orderly manner, and free from mistakes.

Permanence is best secured by making the original computations in a bound book. The size and form may be selected for special purposes; but, for general use, pages $8\frac{1}{2} \times 11$ in. and cross-ruled five divisions to the inch are most convenient.

Orderly arrangement is necessary to permit a proper interpretation of the data, to facilitate the work, and to prevent mistakes.

Just as all field notes show a title, date, party organization, etc., so all computations should show the following data, (1) a title, (2) the date, (3) the names of the computer and checker, and (4) the source of the original data. Also, all steps in the computations should be fully labeled and all checks which are applied should be plainly indicated.

No computations are complete until they have been checked and all mistakes eliminated. Frequently the principal consideration in choosing methods and arrangements of computations is the ease and effectiveness with which checks can be applied. It requires time and effort to check one's work, but nothing except actual dishonesty will discredit one's reputation more than habitual mistakes. Also nothing will contribute toward self-reliance and confidence more than the habit of checking results until they are free from mistakes.

It is difficult for many beginners to appreciate the importance of

the above requirements and to overcome disadvantageous habits that earlier years of practice have established. Accordingly, the student should strive for excellence in this important phase of engineering practice.

7-2. Significant Figures It has been stated that no surveying measurements are exact, since they are all subject to instrumental, personal, and natural errors. The dependent computations, therefore, do not yield exact results; and it is important that the computer shall understand the limitations to which his computations are subject, in order that his results may be consistent with the data from which they are derived.

Accordingly, a distinction must be made between exact and significant figures in computations. Exact figures are those which represent whole units or abstract numbers and are not the result of measurements. Significant figures are those digits in any measured quantity which are known to be correct, including the first doubtful one. An example will illustrate these meanings. Let it be supposed that a distance is measured with an engineer's tape and found to be 67.32 ft. The feet and tenths can be read directly from the divisions on the tape, and the last digit, 2, is estimated. Hence, we may say that there are three certain digits and one doubtful in this number. Therefore, we say that it contains *four* significant figures.

Suppose also that a longer distance is measured with the same tape, with ordinary precision, and is recorded as 2936.47 ft. Now, while the last digit, 7, can be estimated from the tape, it has no meaning and is not significant, because the accumulated errors in measuring the line are such that digit 4 is doubtful. This number, therefore, has four certain digits and one doubtful; and accordingly, it has five significant figures and should be recorded as 2936.5 ft. Further, a distance may be recorded as 5000 ft, but one cannot say how many significant figures this number contains until he knows what precision has been used in the field measurements. If it has been measured with a steel tape so that the total error is not more than 2 or 3 ft, then it has three certain and one doubtful, or four significant figures. If it has been measured by the stadia method so that the total error is perhaps 20 or 30 ft, then it has three significant figures; and if it has been measured by pacing or by odometer so that the total error is 100 or 200 ft, then it has two significant figures only.

The place of the decimal point in a number has no relation to the number of significant figures. Thus, it is possible that the numbers 7.347 ft and 527.4 ft each have four significant figures. The first could be a level rod reading correct to one or two thousandths, and the second could be a taped distance correct to one or two tenths of a foot. Also, the area of a given field may be expressed as 325,600 sq ft or as 7.475 acres, and each has four significant figures. The place of the decimal point is determined merely by the size of the unit in which the result is expressed.

7-3. Consistent Accuracy in Computations In order that computed results shall be consistent with the measurements from which they are derived, it is necessary to note how errors are propagated in the simple arithmetical processes.

Consider the area of a rectangular tract of land of which the two dimensions have been measured as 20.7 ft and 1325.7 ft with an estimated error of ± 0.1 ft in each quantity. The area as computed from the given values is 27,441.99 sq ft, but if each of the quantities is increased by the estimated error of 0.1 ft, then the computed area is 27,576.64 sq ft and the error in the area due to the errors in the sides is 134.65 sq ft. If we express these errors in percentages, they are 0.483%, 0.008% and 0.491% in width, length, and area, respectively. And it is seen that the percentage of error in the area is equal to the *sum* of the percentage of error in the sides, from which the general principle may be stated that *the percentage of error in any product is equal to the sum of the percentages of error in the factors.*

This means that the product of two or more measured quantities cannot be any more accurate (in percentage) than any one of the quantities.

As regards the significant figures in the above quantities, it is noticed that the product of the given dimensions is 27,441.99 sq ft, but that the error in this quantity due to the assumed errors in the sides is 134.65 sq ft. This indicates that the third figure in the product is doubtful, and the other figures have no meaning or significance on account of the errors of measurement. Hence, the product has only three significant figures and should be recorded as 27,400 sq ft. So, although one of the factors has five significant figures, the product has only three, being limited by the number of figures in the width of the tract. Accordingly, the principle stated above in terms

of percentages may be stated in terms of significant figures as follows: the number of significant figures in any product cannot be any greater than the least number of significant figures in any one factor.

This principle is important throughout all engineering computations and should be carefully observed.

7-4. Computation Methods The usual methods of computing include the use of (1) arithmetic, (2) the computing machine, (3) logarithms, (4) the slide rule, and (5) for the special problem of finding areas, the planimeter.

1. *Arithmetic.* Needless to say, a computer should be both rapid and accurate in his arithmetical computations. There are numerous possible short cuts in this work, but with the present availability of computing machines they are not as important as formerly. Only two or three of the most common will be mentioned.

(a) Subtraction is readily checked by noting that the sum of the remainder and subtrahend should equal the minuend.

(b) Any number up to 100 may be squared easily by noting the algebraic relation $a^2 - b^2 = (a + b)(a - b)$, and $a^2 = (a + b)(a - b) + b^2$. Example. Find the square of 93. Solution. Let $a = 93$ and let $b = 3$, which when subtracted from a, renders the last digit zero. Then $93^2 = 96 \times 90 + 9 = 8640 + 9 = 8649$.

(c) Any number can be easily multiplied by 25 if it is divided by 4 and the quotient multiplied by 100. Example. $716 \times 25 = (716/4) \times 100 = 17{,}900$. Also any number can be easily divided by 25 if it is multiplied by 4 and the product divided by 100. Example, $13{,}276 \div 25 = 13{,}276 \times 4 \div 100 = 531.04$.

2. *Computing Machines.* The use of a computing machine is an advantage as much in preventing mistakes as in the time and effort it saves. Because of the ease with which the operations are made, particular attention should be paid to the matter of significant figures in final results.

3. *Logarithms.* Since many surveying measurements are made to four or more significant figures, slide rules are not sufficiently accurate; and in a series of related measurements the arithmetical computations become laborious. In such cases logarithms are used to facilitate the work and to avoid mistakes. They are often used to check computations made by natural numbers.

Four places of logarithms yield four places in the corresponding numbers, five places yield five numbers, etc., and, referring to Art.

6-5 regarding the relation between the accuracy of angles and related distances, it may be noted that distances measured with an accuracy of four significant figures correspond to angles measured to about the nearest 02′ or 03′. Also, distances measured to five significant figures correspond to angles measured to about the nearest 20″. Accordingly, it may be said that for surveys of fair accuracy where distances are measured with an accuracy of about 1/1000, and angles to the nearest 02′ or 03′, four places of logarithms are sufficient. And for surveys of good accuracy where the errors in measured distances and angles are from 1/1000 to 1/10,000 and angles from 02′ to 20″, five places of logarithms are suitable.

4. *The Slide Rule.* The slide rule is useful for many subordinate operations in surveying computations, but for the most part four or five significant figures are required, for which the ordinary 10-in. slide rule is not suitable. It does, however, provide a rough check on all computations.

7-5. Latitudes and Departures As used in connection with surveying computations, the terms *latitude* and *departure* may be defined as follows:

The latitude of a course is its projection on the reference meridian; i.e., it is the distance one end of the course is north or south of the other end.

The departure of a course is its projection on the east-west line perpendicular to the reference meridian; i.e., it is the distance one end of the line is east or west of the other end.

The magnitude of the latitude of a course is the product of its length by the cosine of its bearing.

The magnitude of the departure of a course is the product of its length by the sine of its bearing.

The computations are made with natural numbers and a computing machine if one is available; otherwise, logarithms are used. The work may be arranged as shown in Fig. 7-1.

The data for these computations are taken from the azimuth traverse notes of Fig. 6-2, and comprise the measured distances and calculated bearings, as transcribed in columns 1, 2, and 3 of Fig. 7-1.

TRAVERSE COMPUTATIONS

7-6. Steps in Calculation of Traverse The general sequence of computing a closed traverse, leading up to the determination of the

Computed by A.F. Sears
Data from Field Book p 10. May 20, 19

AREA OF FIELD

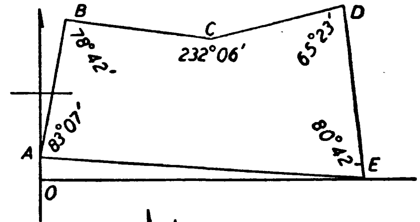
Course	Bearing	Distance	Latitudes		Departures		Balanced Latitudes		Balanced Departures		Coordinates N	Coordinates E	Pts.
			N	S	E	W	N	S	E	W			
A B	N 9° 30'E	575.4	567.5		95.0		567.6		94.9		51.4	0.0	A
B C	S 69° 12'E	495.3		175.9	463.0			175.8	463.0		619.0	94.9	B
C D	N 58° 42'E	558 7	290.2		477.4		290.3		477.3		443.2	557.9	C
D E	S 6° 41'E	738.7		733.7	86.0			733.5	85.9		733.5	1035.2	D
E A	N 87° 23'W	1122.1	51.2			1120.9	51.4			1121.1	0.0	1121.1	E
	Sum	3490.2	908.9	909.6	1121.4	1120.9	909.3	909.3	1121.1	1121.1			
			−0.7 ft		+0.5 ft								

$$E_T = \sqrt{\overline{0.7}^2 + \overline{0.5}^2} = 0.86 \text{ ft} \qquad \frac{E_T}{\text{Perimeter}} = \frac{1}{4100}$$

Area = 561,200 sq ft
= 12.88 Acres

LOGARITHMS

	A B	B C	C D	D E	E A
Lat.	567.50	175.89	290.25	733.68	51.22
Log Lat	2.75397	2.24523	2.46278	2.86551	1.70951
Log Cos. Bear.	9.99400	9.55036	9.71560	9.99704	8.65947
Log Dist.	2.75997	2.69487	2.74718	2.86847	3.05004
Log Sin. Bear.	9.21761	9.97073	9.93169	9.06589	9.99955
Log Dep.	1.97758	2.66560	2.67887	1.93436	3.04959
Dep.	94.97	463.02	477.39	85.97	1120.9



NATURAL NUMBERS

Course	Bearing	Distance	Sin. Bear.	Cos Bear.	Latitudes	Departures
A B	N 9°30'E	575.4	.16505	.98629	N 567.5	E 95.0
B C	S 69°12'E	495.3	.93483	.35511	S 175.9	E 463.0
C D	N 53°42'E	558.7	.85446	.51952	N 290.2	E 477.4
D E	S 6°41'E	738.7	.11638	.99320	S 733.7	E 86.0
E A	N 87°23'W	1122.1	.99896	.04565	N 51.2	W 1120.9

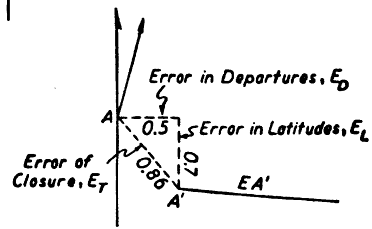


FIG. 7-1. Data for Area of Field.

enclosed area, is as follows: (1) distribute the angular error of closure and calculate the bearings, (2) compute the latitudes and departures, (3) find the total error of closure and balance the survey, (4) compute the coordinates and the area, and (5) check the result. These steps will now be explained and demonstrated by computing the area of the five-sided azimuth traverse of Fig. 6-2.

7-7. Errors of Closure and Balancing the Survey For a closed field, the sums of the north and south latitudes should be equal, and the sums of the east and west departures should be equal. While this condition is true in theory, because of the inherent errors in the measurements of the distances and angles, it seldom is true in practice. The magnitude of the difference between these sums is an indication of the precision of the measurements as regards the accidental errors. Any systematic error present, e.g., the incorrect length of tape, would have the effect of causing an error in the area, but since all of the courses would be affected proportionally to their length, this source of error would have no effect on the error of closure.

The relation between these errors and the total error of closure is shown in Fig. 7-1, from which it is evident that the *total error of closure*, E_T, is the hypotenuse of a right triangle of which the errors in latitudes, E_L, and in departures, E_D, are the two sides.

Hence, in this example $E_T = \sqrt{0.7^2 + 0.5^2} = 0.86$ ft.

This error is also expressed as a ratio to the perimeter, in which the numerator is unity. The result is given in round numbers only. It is then termed the *relative error* of closure. Thus,

$$E_T = \frac{0.86}{3490} = \frac{1}{4100}$$

Before computing the area of a tract it is desirable to balance the values of the latitudes and departures so that the sums of the north and south latitudes are equal, and the sums of the east and west departures are equal, and thus the field is closed mathematically as it is in fact. This process is termed *balancing the survey*.

This problem of adjusting the observations has received much attention and several methods have been devised. Two of these will now be explained. The first may be called the *field-condition* method and is based on a knowledge of the field conditions and the judgment of the engineer. For example, it may have been that side BC in

the field of Fig. 7-1 was difficult to observe because of obstructions, and hence in adjusting the angular error the corrections would be applied to angles B and C. Also, side EA might have been difficult to measure because of rough ground; and, since this side has its direction in an east-west direction, the correction in departures would be applied principally to the departure of this course. And so by arbitrary adjustments based on the field conditions, the survey is balanced.

The second method makes use of a rule called the *compass rule* which is to be used when it is assumed that the errors are evenly distributed throughout the survey. It may be stated as follows: *The correction to the latitude (or departure) of a course is to the total error in latitudes (or departures), as the length of the course is to the perimeter of the field.*

The computation for the correction to the latitude of the course AB is, therefore,

$$\frac{C_l}{0.7} = \frac{570}{3500} \quad \text{or} \quad C_l = 0.11 \text{ ft}$$

and for the correction to the departure of the course,

$$\frac{C_d}{0.5} = \frac{570}{3500} \quad \text{or} \quad C_d = 0.08 \text{ ft}$$

These computations are made conveniently with the slide rule; and, when applied to the nearest tenth of a foot only, the results are as shown in the columns headed "balanced" latitudes and departures.

A so-called *transit* rule and other *modified* compass and transit rules have been proposed, but generally these rules do not accomplish the results intended. It is not within the scope of this book to discuss this subject in detail, but it can be asserted with assurance that, if used according to the stated conditions, the two rules given above will be satisfactory for all areas except rectangular tracts whose sides have the approximate directions of north, south, east and west.

7-8. Computation of Coordinates A brief review of the theory of rectangular coordinates will aid in understanding the methods to be described in the articles that follow.

All coordinate values are computed from an origin fixed by the intersection of an X-axis and a Y-axis, and in surveying theory the

X-axis is assigned the east-west direction and the Y-axis the north-south direction. Also, the north and east directions are given the positive sign, and the south and west directions are given the negative sign. With regard to the result obtained by computing an area by the method of coordinates, the location of the origin is imma-

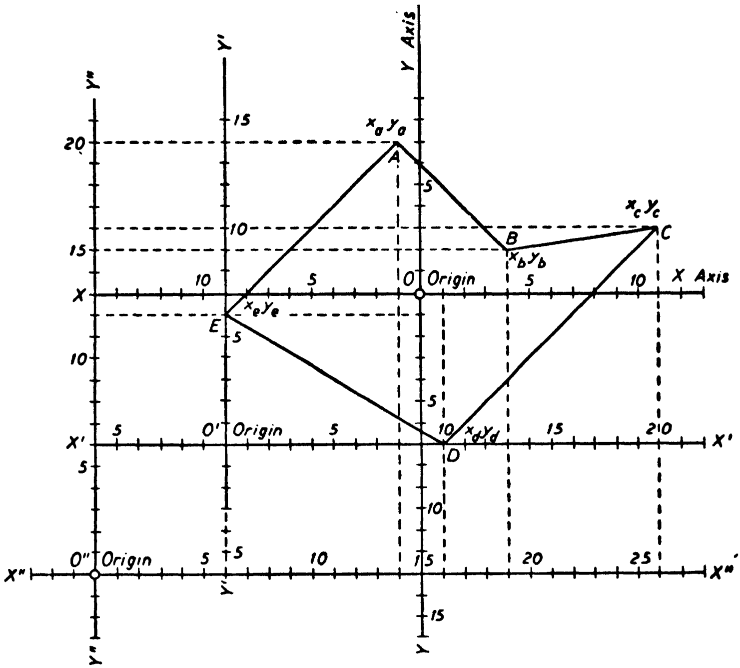


FIG. 7-2. Relations Between Different Origins.

Pts.	Origin O		Origin O'		Origin O"	
	x	y	x'	y'	x''	y''
A	− 1	+ 7	+ 8	+14	+14	+20
B	+ 4	+ 2	+13	+ 9	+19	+15
C	+11	+ 3	+20	+10	+26	+16
D	+ 1	− 7	+10	0	+16	+ 6
E	− 9	− 1	0	+ 6	+ 6	+12

Table of Coordinates For Different Origins

terial, but for given conditions it is desirable to choose the position of the origin such that certain advantages may be derived. Some of these considerations are discussed with the aid of Fig. 7-2 and the Table of Coordinates.

In Fig. 7-2 consider first the origin O in its relation to the figure $ABCDEA$. If a scale of units is assigned and the signs for the directions are observed, the values of the coordinates of the vertices will be as shown in the table.

Also, it may be supposed that origin O' is chosen so that the Y'-axis passes through the most westerly point, E, and the X'-axis passes through the most southerly point, D. It is now evident that the position of the O'-origin is 9 units west and 7 units south of the O-origin. Accordingly, the values of the coordinates of the figure with respect to the O'-origin will be found by adding 9 units to all x-values and 7 units to all y-values in the O-origin series, as shown in the O'-series in the table. It may be noticed that this position of the O'-origin has given the positive sign to all of the coordinate values and has made the y-value of D and the x-value of E equal to zero. This condition simplifies all subsequent computations using these coordinates.

Again, the origin may be moved still farther away from the figure to another position O''. Now it is evident that the O''-origin is 15 units west and 13 units south of the O-origin, and the values of the coordinates with respect to that origin will be found by adding 15 units to all x-values and 13 units to all y-values in the O-origin series. Again, all the coordinate values have the positive sign and are increased in magnitude. This location of the origin is sometimes desirable so that the field survey may be extended (within limits) in any direction without encountering any negative signs in the coordinates.

For the computation of land areas, the position of the origin is nearly always chosen as shown, either for the O'- or for the O''-origin.

7-9. Plane Coordinate Systems Engineers and surveyors have used for many years in their work a variety of unrelated and arbitrary plane coordinate systems. Frequently, different systems of this type are utilized in the same community or even at the same industrial site. Such coordinate systems are defined by assigning x and y values to a chosen survey point and taking either an assumed meridian or the true meridian through the initial point as grid north. These systems lack official recognition, cannot be correlated with other surveys, and pose problems associated with the convergency of the true meridians when extended over a large area.

It is much better, whenever possible, to correlate local traverse and triangulation surveys with the national network of horizontal control established by the U.S. Coast and Geodetic Survey and to use the appropriate state plane coordinate system designed by that agency for each state. This means that the engineer executing local surveys should attempt to reference his survey to government traverse and triangulation points whose x and y state plane coordinates are accurately known. The resulting calculated coordinate positions in the local survey will then have much more significance and it will be possible to relate such a survey to any other contiguous one which was likewise computed in terms of state plane coordinates. Further discussion of this subject is beyond the scope of this text.

AREA COMPUTATIONS

7-10. Computation of Areas by Triangles The area of any field can be found by dividing it up into a series of triangles, making the necessary measurements and then calculating the area by any of the usual trigonometrical formulas. This method is excellent for small areas with few sides, but for larger areas with many sides the method of coordinates is more satisfactory.

7-11. Computation of Areas by Coordinates The mathematical theory of coordinates provides a method for computing the area of any closed figure bounded by straight lines. The method has many applications in surveying practice, an important one of which is to compute the area of a tract of land whose boundary has been measured by a compass or a transit traverse.

An alternate method for computing land areas that has had much usage makes use of values called "*double meridian distances,*" which method, after all, is nothing more than a special adaptation of the method of coordinates. The method of coordinates however, is more generally applicable, is more easily understood and applied, and the same computations supply the data for the simple and accurate coordinate method for plotting the tract.

From mathematics we have the rule that, taking the vertices in order around a closed figure, the area is equal to *one half the sum of the products of each ordinate multiplied by the difference between the two adjacent abscissas, always subtracting the preceding from the following abscissa.*

In applying this rule to land-survey computations, the terms *ordi-*

nate and *abscissa* are supplanted by the corresponding coordinates *North* and *East*. With these substitutions made, using the letters N and E to indicate the coordinates, the rule may be applied by means of the following arrangement. The coordinates for each vertex are written in the form of a fraction, of which the numerator is the ordinate (N) and the denominator is the abscissa (E). Also, the series of fractions thus written is enclosed between vertical dashed lines. Now the first numerator N_1 is to be multiplied by the difference between the two adjacent denominators, E_2 and E_5, always subtracting the preceding abscissa E_5 from the following E_2. To indicate this operation the denominator of the last fraction to the right, E_5, is written outside the dashed line to the left of the first fraction. Likewise, the denominator of the first fraction E_1 is written outside of the dashed line to the right of the last fraction. The completed arrangement follows:

$$\frac{\quad}{E_5} \;\vdots\; \frac{N_1}{E_1} \quad \frac{N_2}{E_2} \quad \frac{N_3}{E_3} \quad \frac{N_4}{E_4} \quad \frac{N_5}{E_5} \;\vdots\; \frac{\quad}{E_1}$$

The area is now given by the equation:

$$A = \tfrac{1}{2}[N_1(E_2 - E_5) + N_2(E_3 - E_1) + N_3(E_4 - E_2) + N_4(E_5 - E_3) + N_5(E_1 - E_4)]$$

For the example given above, the fractions and computations are as follows:

$$\frac{\quad}{1121.1} \;\vdots\; \frac{51.4}{0.0} \quad \frac{619.0}{94.9} \quad \frac{443.2}{557.9} \quad \frac{733.5}{1035.2} \quad \frac{0.0}{1121.1} \;\vdots\; \frac{\quad}{0.0}$$

		+	−
51.4 ×	(−1026.2) =		52,746
619.0 ×	557.9 =	345,340	
443.2 ×	940.3 =	416,740	
733.5 ×	563.2 =	413,107	
0.0 ×	(−1035.2) =	0.0	0.0
		1,175,187	52,746
		52,746	
	2)	1,122,441	
		561,220 sq ft = 12.88 acres	

Also, the above rule may be stated as follows: the area is equal to one half the sum of the products of each abscissa multiplied by the difference between the two adjacent ordinates, always subtract-

ing the preceding from the following ordinate. The form and computation follow:

$$
\begin{array}{c|ccccc|c}
\underline{N_5} & \dfrac{N_1}{E_1} & \dfrac{N_2}{E_2} & \dfrac{N_3}{E_3} & \dfrac{N_4}{E_4} & \dfrac{N_5}{E_5} & \underline{N_1} \\[2ex]
\underline{0.0} & \dfrac{51.4}{0.0} & \dfrac{619.0}{94.9} & \dfrac{443.2}{557.9} & \dfrac{733.5}{1035.2} & \dfrac{0.0}{1121.1} & \underline{51.4}
\end{array}
$$

$$
\begin{array}{rcrcr|r}
 & & & & + & - \\
0.0 & \times & 619.0 & = & 0.0 & 0.0 \\
94.9 & \times & 391.8 & = & 37{,}182 & \\
557.9 & \times & 114.5 & = & 63{,}880 & \\
1035.2 & \times & -443.2 & = & & 458{,}801 \\
1121.1 & \times & -682.1 & = & & 764{,}702 \\
\hline
 & & & & 101{,}062 & 1{,}223{,}503 \\
 & & & & & 101{,}062 \\
\hline
 & & & & & 2)1{,}122{,}441 \\
\hline
 & & & & & 561{,}220 \text{ sq ft} = 12.88 \text{ acres}
\end{array}
$$

It is necessary, of course, to observe the algebraic signs carefully, and it may be noticed that for one of the above computations the area has a positive, and for the other, a negative sign. But the sign of the result is immaterial if due regard is paid to the signs in the computations.

7-12. Checks The checks which apply to the above computations are as follows:

1. Assuming that the field work is correct, the magnitude of the error of closure provides a check on the calculation of the latitudes and departures. It should be added that this value also provides a check on the accidental errors of the field work including both angles and distances. Accordingly, an error of closure larger than that permitted may be occasioned either by incorrect field measurements or by mistakes in the computations. If the computations are proved to be correct, the error must be found in the field work.
2. The calculation of the coordinates is checked by the condition that the last calculated abscissa is equal numerically to the departure of the last course, with opposite sign. Likewise the last calculated ordinate is equal numerically to the latitude of the last course, with opposite sign.
3. The calculation of the area is checked by using both arrangements of the coordinates given above.

7-13. Double Meridian Distances The method of computing areas by the use of double meridian distances has been mentioned in Art. 7-11. This method will now be described, referring to the computations and sketch of Figs. 7-3 and 7-4.

The latitudes and departures of the various courses are computed and adjusted as described in Art. 7-7. The results for a typical example are shown in Fig. 7-3.

			Latitudes		Departures		D.M.D.	Double Areas	
Course	Bearing	Distance	N	S	E	W		+	−
A B	N 57°13'E	695.44	376.55		584.67		584.67	220157	
B C	S 64°53'E	675.38		286.67	611.52		1780.86		510519
C D	S 4°42'W	682.70		680.41		55.94	2336.44		1589737
D E	S 50°04'W	348.58		223.75		267.29	2013.21		450456
E F	N 75°12'W	741.14	189.31			716.55	1029.37•	194870	
F A	N 14°03'W	644.25	624.97			156.41	156.41	97752	
								512779	2550712
									512779
								2	2037933
								Area =	1,018,966 sq ft

FIG. 7-3. Data for Double Meridian Distances.

The *meridian distance* of any side may be defined as the distance from its mid-point to the reference meridian. Thus, the meridian distance of the course AB is the distance OG; and for the course BC it is NH, etc. Obviously, the *double meridian distance* of any course is twice its meridian distance.

The double meridian distances for the succeeding sides of the traverse may be computed as follows: the double meridian distance of $AB = 2 \times OG = OG + PQ =$ the departure of AB; the double meridian distance of $BC = 2 \times NH = (2 \times OG) + (2 \times PQ) + (2 \times QH)$; these quantities are the double meridian distance of AB, the departure of AB, and the departure of BC, respectively. The double meridian distance of $CD = 2 \times RI = (2 \times RS) + (2 \times ST) - (2 \times TI)$; and these quantities are the D.M.D. of BC, the departure of BC, and the departure of CD. The last distance, being west, has the minus sign.

According to the computations just indicated, the general rule may be stated: *The D.M.D. of any course is equal to the D.M.D. of the preceding course, plus the departure of that course, plus the departure of the course itself.*

Applying this rule to the data of Fig. 7-3, the D.M.Ds. of the suc-

cessive courses are computed. It will be noticed that the D.M.D. of the last course, *FA*, is equal to the departure of that course, but with the opposite sign. This condition provides a check upon the computation of all the D.M.Ds.

7-14. Areas by D.M.Ds. The use to be made of the D.M.Ds. is to find the area of the tract. This is done by multiplying the D.M.D. of each course by the latitude of that course, giving due regard to signs; by adding the products algebraically and dividing the sum by two. Thus, each D.M.D. is multiplied by the latitude of that course, the double areas being recorded in the + or − columns corresponding to a north or a south latitude, respectively. The area is then one half of the algebraic sum of the double areas.

The double areas have a graphical interpretation, shown in Fig. 7-4. The first double area is represented by double the area *AMB*;

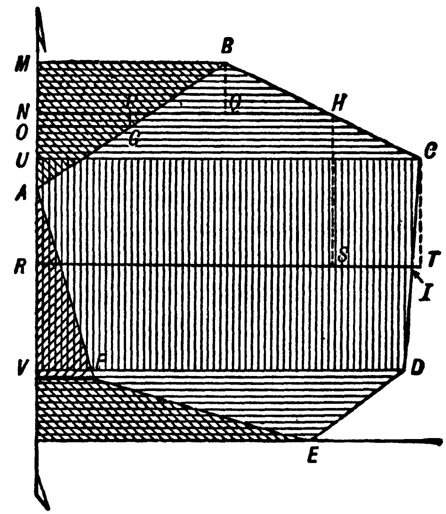


FIG. 7-4. Areas by D.M.D.s.

the second double area is represented by double the area *MBCU*; the third double area is twice *UCDV*, etc. It may now be noticed that areas corresponding to courses for which the latitude has a plus sign lie to the left and outside of the tract; whereas the areas corresponding to courses for which the latitude has a minus sign include both the area within and without the tract. Accordingly, when the three

double areas corresponding to the courses AB, EF, and FA are subtracted from the double areas corresponding to the courses BC, CD, and DE, the remainder is the double area within the boundary of the tract.

7-15. Double Parallel Distances By a strictly analogous procedure, double parallel distances can be computed, using the latitudes of the successive courses instead of the departures, and using the east line through the most southerly point as the reference parallel.

Likewise, the area can be found by using the D.P.Ds. and the corresponding departures of the courses. If these computations should be made, the resulting area should equal that found by the D.M.Ds. and thus provide a complete check on the computations.

7-16. Areas with Irregular Boundaries Areas with irregular or curved boundaries are usually measured by establishing a baseline conveniently near and by taking offsets at regular intervals from the baseline to the boundary. Three methods are most commonly used, namely, (1) the trapezoidal method, (2) Simpson's one-third rule, and (3) the coordinate method.

Figures 7-5, a and b, represent two types of irregular areas, the first with an irregular boundary and the second bounded by the circular curve of a street or a roadway property line. Offsets h_1, h_2, etc., have been measured from the base line to the boundary at regular intervals, b.

1. *Trapezoidal Method.* If the ends of the offsets in the boundary line are assumed to be connected by straight lines, a series of trapezoids is formed, the bases being the offsets and the altitudes being the common distance b. Accordingly, the area of the first trapezoid is $\frac{b(h_1 + h_2)}{2}$, of the second it is $\frac{b(h_2 + h_3)}{2}$, etc. Summing these up, we have for the total area, A, the following equation, in which n equals the number of offsets:

$$A = b\left[\frac{h_1 + h_n}{2} + (h_2 + h_3 + \ldots. + h_{n-1})\right]$$

Example: Find the area of Fig. 7-5a if the common interval is 25

ft and if the offsets are 29.6, 28.2, 34.3, 41.5, 39.6, 27.2, and 18.4 ft, respectively.

$$A = 25\left(\frac{29.6 + 18.4}{2} + 28.2 + 34.3 + 41.5 + 39.6 + 27.2\right)$$

$$= 4870 \text{ sq ft}$$

2. *Simpson's One-Third Rule.* Simpson's one-third rule may be applied to areas similar to those illustrated in Fig. 7-5, *a* and *b*,

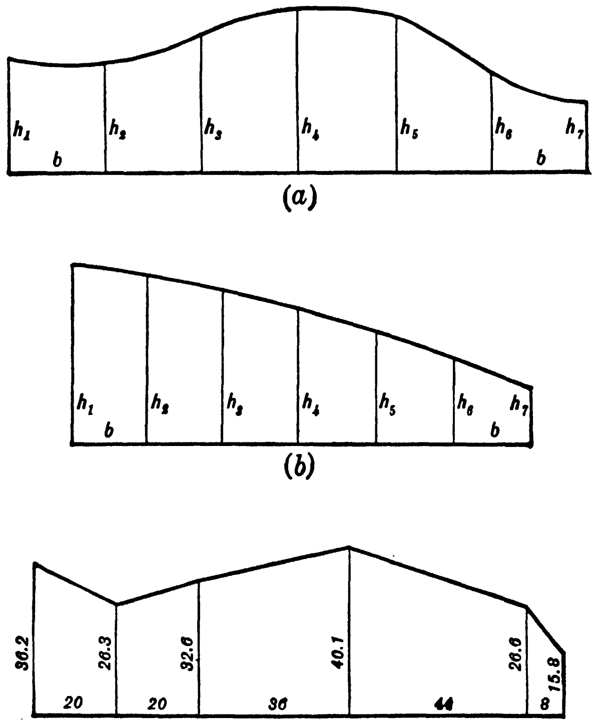


FIG. 7-5. Irregular Boundaries.

where the offsets have a common interval, *b*, and provided an odd number of offsets are taken. The rule may be stated as follows: *The area is equal to one-third of the common interval between offsets, multiplied by the sum of the first and last offsets, plus two times the*

sum of the other odd offsets, plus four times the sum of the even offsets; or, if n equals the number of offsets,

$$A = \frac{b}{3}\Big[h_1 + h_n + 2(h_3 + h_5 + \dots + h_{n-2}) + 4(h_2 + h_4 + \dots + h_{n-1})\Big]$$

This rule is based on the assumption that the curve passing through the ends of the first three offsets is a parabola; likewise, the curve through the ends of offsets, 3, 4, and 5, also through the ends of offsets 5, 6, and 7, etc. And it is supposed that this series of parabolic curves will approximate the boundary line more closely than do straight lines, and so yield a more accurate value for the area.

EXAMPLE. Find the area of Fig. 7-5*b* if the common interval is 20 ft and the offsets are 44.3, 42.0, 39.4, 28.7, 22.3, and 14.6, respectively.

Since there is an odd number of offsets, the rule can be applied to the entire area as follows:

$$A = \frac{20}{3}\Big[44.3 + 14.6 + 2(39.4 + 28.7) + 4(42.0 + 34.6 + 22.3)\Big]$$
$$A = 3938 \text{ sq ft}$$

3. *The Coordinate Method.* If the boundary of an area is such that offsets are best taken at irregular intervals, as shown in Fig. 7-5*c*, the area may be calculated as a series of separate trapezoids, or by the coordinate method. The latter method has been explained in previous articles.

EXAMPLE. Find the area of Fig 7-5*c* with dimensions in feet as shown. Taking the origin at the foot of the left-hand offset, the arrangement of coordinates and computation are as follows:

$$\frac{\quad}{128.0} \;\vdots\; \frac{0}{0} \quad \frac{36.2}{0} \quad \frac{26.3}{20.0} \quad \frac{32.6}{40.0} \quad \frac{40.1}{76.0} \quad \frac{26.6}{120.0} \quad \frac{15.8}{128.0} \quad \frac{0}{128.0} \;\vdots\; \frac{\;}{0}$$

$$\begin{aligned}
36.2 \times 20.0 &= 724. \\
26.3 \times 40.0 &= 1052. \\
32.6 \times 56.0 &= 1825.6 \\
40.1 \times 80.0 &= 3208.0 \\
26.6 \times 52.0 &= 1383.2 \\
15.8 \times 8.0 &= 126.4 \\
2)&\overline{8319.2} \\
&4159.6
\end{aligned}$$

Check:

	+	−
20.0 × − 3.6 =		72.0
40.0 × +13.8 =	552.0	
76.0 × − 6.0 =		456.
120.0 × −24.3 =		2916.
128.0 × −26.6 =		3404.8
128.0 × −15.8 =		2022.4
	552.0	8871.2
		552.0
		2)8319.2
		4159.6

The relative merits of the above methods may be compared as follows:

1. The trapezoidal method is simplest and is sufficiently accurate for most areas, provided proper care is taken in the measurements. It may be noted (Fig. 7-5*a*) that the calculated area will be slightly too large for trapezoids where the boundary line is concave upwards, and too small for trapezoids where the line is concave downwards. Thus, this method is more accurate where the boundary line consists of segments of contrary flexure.

2. Simpson's rule is more laborious to apply than the trapezoidal rule, but it is more accurate for all conditions. It is especially applicable to figures like that shown in Fig. 7-5*b*.

3. The coordinate method is best for offsets at irregular intervals.

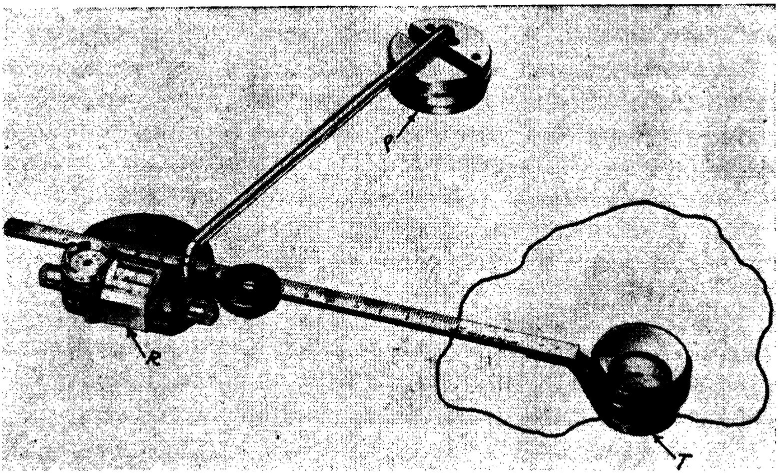
7-17. Areas by Polar Planimeter The planimeter is an instrument by means of which the area of a plotted, closed figure may be determined directly by tracing the perimeter and reading the result from the scale.

The polar type of this instrument is illustrated in Fig. 7-6. Its essential features are an anchor point or pole P, a tracing point T, and a roller R, which has a graduated scale on a drum.

The two arms connecting these points are movable about the connecting pivot. In one type, the tracing arm is of fixed length and hence is capable of reading areas in one unit only, usually square inches. In another type, Fig. 7-6, the tracing arm is adjustable in a sleeve, to read areas in different units, corresponding to the scale of the map. The arms are usually adjusted so that one revolution of the roller measures an area of 10 sq in.; the scale is divided into 100

parts, and the vernier reads to 1/10 of a scale division, or 1/1000 of 10 sq in. = 0.01 sq in.

With some instruments there is provided a flat metal bar with a needle point at one end and a hole drilled through the bar at a dis-



Courtesy of National Blue Print Co.

FIG. 7-6. Optical Polar Planimeter.

tance from the needle point equal to the radius of a circle of 10 sq in. area. With the needle point pressed into the drawing paper and the tracing point inserted in the hole, an area of 10 sq in. can be quickly and accurately traced, and, if necessary, the tracing arm can be precisely adjusted. If the instrument is of the fixed-arm type, the tracing arm cannot be adjusted, but a constant can be determined by which all results are to be multiplied, to determine correct values.

If a proving bar, described above, is not provided, a square of known area can be carefully drawn and the perimeter traced a few times, either to adjust the tracing arm or to find the instrument's constant.

It can be shown by the calculus that the amount the roller turns is a measure of the area included within the perimeter traced. Thus, the instrument is a mechanical integrator and, in fact, is sometimes used to integrate plotted mathematical equations.

In use, the pole is placed in any convenient position outside the area and the tracing point is placed at some initial point of the per-

imeter to be traced. The scale is then read by means of the index and vernier provided on the roller frame. This reading of the scale is recorded as the initial reading. The tracing point is then moved carefully around the perimeter, during which process the roller will both turn and slide, until the initial point is again reached. The scale is again read, and recorded as the final reading. The difference between the two readings is a measure of the area within the perimeter traced. If the direction of the tracing point has been clockwise, the result is positive, and, if counterclockwise, it will be negative. Two or more determinations of each area should be made to provide a check and to secure a more accurate result.

If the area is too large to be included by the tracing area for one position of the pole, the area can be divided into the requisite number of subdivisions.

Areas are often desired on cross-section or profile paper on which the horizontal and vertical scales are not the same; but this causes no difficulty, since such areas are always proportional to the product of horizontal and vertical dimensions.

The drawing paper should be smooth and free from wrinkles and large enough that the roller will not need to pass across the edge of the sheet. The bearings of the instrument should be adjusted so as to permit free movement, but without any play.

When carefully manipulated, planimeter results are surprisingly accurate. The percentage of error decreases as the size of the area increases, one reason being, of course, that the area increases with the square of the perimeter. The mean of two determinations should be correct within 1% for small areas and may easily be within 0.1% for areas of 20 sq in. or more. This precision is sufficient for such purposes as determining drainage areas, cross-section and contour areas for earthwork, and reservoir areas and volumes. Its precision and facility in operation make this a most useful instrument for determining plotted areas of any shape whatsoever.

VOLUME COMPUTATIONS

7-18. Volumes by Average End Areas In engineering projects, volumetric quantities are usually determined by finding the areas of parallel sections and multiplying their mean value by the perpendicular distance between them. This procedure is called the method of *average end areas.*

The methods of determining the areas have been indicated in the preceding articles, and for contours, in Art. 10-6.

The most common use of this method is for finding earthwork volumes for the construction of highways, railways, drainage ditches, canals, etc. In such projects the cross-section areas are usually determined at intervals of 100 ft. The volume for such a 100-ft section is given by the equation

$$V = \frac{100 \times (A_1 + A_2)}{27 \times 2} = 1.85A$$

in which V is the volume in cubic yards, A_1 and A_2 are the areas of the end sections in square feet, and A is the sum of the two end sections.

If the ground is uneven or changes slope abruptly, intermediate sections are taken such that the errors in resulting volumes will not be serious. Theoretically, this method is not exact unless the two end areas are equal, but except as indicated in the following article, the resulting errors are insignificant.

7-19. Volumes of Prismoids When the ground surface is such that the two end areas are widely different in area, or when high precision is desired as in the case of rock quantities in excavation, or of volumes of concrete structures, the average-end-area method is not sufficiently exact and the volumes are calculated as prismoids.

A prismoid may be defined as a solid having parallel, plane bases, and having sides which are plane surfaces. Its volume is given by the equation

$$V = \frac{h}{6}(A_1 + 4A_m + A_2)$$

in which V is the volume in cubic feet; h is the perpendicular distance between the bases; A_1 and A_2 are the end sections; and A_m is the area of a mid-section.

It should be noted that the area of a mid-section will not ordinarily be the same as the mean of the two end areas, but must be computed from the linear dimensions which are mean between those of the end sections.

The application of the prismoidal formula to earthwork sections is somewhat laborious since the mean dimensions of the end sections must be found before the area of the mid-section can be computed.

These computations are simplified by computing a correction to be applied to the volume as calculated by the average-end-area method of the previous article. This correction is $C_v = 0.31\ (C_1 - C_2)\ (W_2 - W_1)$, in which C_v is the prismoidal correction in cubic yards for an earthwork section 100 ft long, C_1 and C_2 are the center cuts or fills at the end sections A_1 and A_2, and W_1 and W_2 are the corresponding distances between slope stakes at these sections. The prismoidal correction C_v, is to be subtracted from the average-end-area volume.

7-20. Borrow Pits In constructing the earthwork for roadways, levees, etc., it is frequently necessary to excavate material from areas adjacent to the project to form the embankments. Such excavations are called *borrow pits.* They may be quite irregular in shape, and the volume of material is usually determined by first dividing the area into squares of suitable size and then taking level readings at the corners of the squares both before and after the work of excavation. From these data the volume may be computed; the volume removed from any square being calculated as the average of the heights at the four corners times the area. Some corners are common to more than one square, thus d, in Fig. 7-7, is common to one, e is common to two, j is common to three, and f is common to four squares. Hence, the work may be simplified somewhat if each corner height is multiplied by the number of squares to which it is common, and the sum taken and averaged, to find the average cut of the whole area. By this method the volume included within a group of squares is given by the following squation:

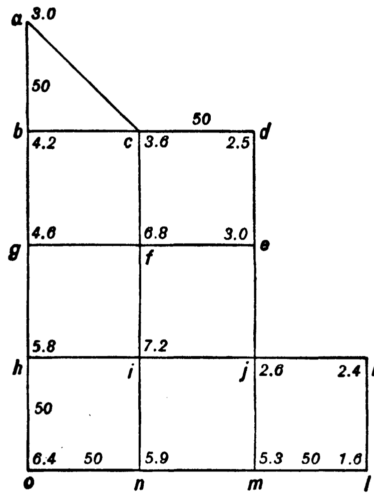


FIG. 7-7. Borrow Pit.

$$V = \frac{A}{4 \times 27} (h_1 + 2h_2 + 3h_3 + 4h_4)$$

in which A is the area of one square in square feet, h_1, h_2, h_3, and h_4

are the corner heights common to one, two, three, and four squares, respectively.

The total area may include additional triangles or trapezoids, which must be computed separately.

The volume indicated in Fig. 7-7 may be computed as follows:

h_1	h_2	h_3	h_4
$b = 4.2$	$c = 3.6$		$f = 6.8$
$o = 6.4$	$g = 4.6$	$j = 2.6$	$i = 7.2$
$l = 1.6$	$h = 5.8$		
$k = 2.4$	$n = 5.9$		
$d = 2.5$	$m = 5.3$		
	$e = 3.0$		
17.1	28.2	2.6	14.0

$$V_1 = \frac{2500}{4 \times 27} [17.1 + (2 \times 28.2) + (3 \times 2.6) + (4 \times 14.0)] = 3178$$

$$V_2 = abc = \frac{2500}{2 \times 3 \times 27} (10.8) = 167$$

$$3345 \text{ cubic yards}$$

MISCELLANEOUS COMPUTATIONS

7-21. Traverse With Missing Parts Sometimes the field conditions prevent the measurement of one or more distances or angles of a traverse, but any two missing parts of such a traverse can be supplied and the area calculated, as may now be described. The missing data may be the length and the direction of any one side, or they may be the length of one side and the bearing of another. A disadvantage of the method is that no check on the field measurements is provided.

One Side.—If the dimensions of one side only are missing, it can be considered to be the closing line of the traverse, and its length and bearing can be computed just like the error of closure of a complete traverse (see Art. 7-7). For example, suppose that the length and the bearing of side DE, Fig. 7-8, could not be measured. Then the latitudes and departures of the other sides are computed and the sums taken. Evidently, the difference between the east and west departures (267.29) will be the departure of side DE; also, the difference between the north and south latitudes (223.75) will be the latitude of side DE. Hence, the bearing of DE is $\tan^{-1} 267.29/223.75 = 50°04'$, or S 50°04′ W; and the length of DE is $223.75/\cos 50°04' = 348.58$.

Two Adjacent Sides.—Let it be supposed that the two missing dimensions affect two adjacent sides, as, for example, the bearing of side DE and the length of side EF. Now the sums of the latitudes and departures of the remaining sides are taken and the difference

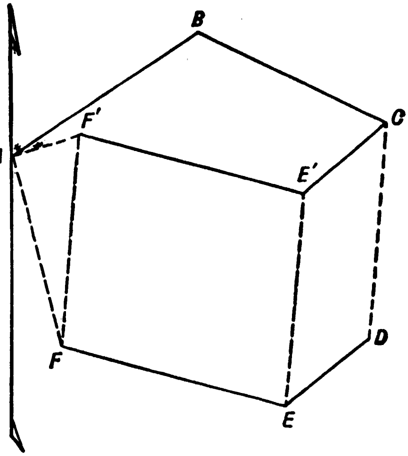


FIG. 7-8. Missing Parts.

in the departures is the departure of the assumed closing side DF; likewise, the difference in the latitudes is the latitude of DF. As explained in the previous paragraph, the length and bearing of side DF are now computed. In triangle DEF two sides and one angle are known; hence, the remaining angles and side can be found and thus the missing parts are supplied.

Two Nonadjacent Sides.—If the missing data affect two sides that are not adjacent, the following procedure may be used. In Fig. 7-8 it may be supposed that the bearing of CD and the length of FA are missing. A solution is now possible if the known sides FE and ED are shifted, without changing their bearings, adjacent to the other known sides AB and BC. Thus lines FE and ED are shifted to the position $F'E'$ and $E'C$, sides FF' and EE' being parallel and equal to side DC. Now the known sides, AB, BC, CE', $E'F'$, form a figure for which $F'A$ may be taken as the closing line. As previously explained, the bearing and length of side $F'A$ can now be computed. In triangle AFF' it is apparent that the only unknown values are the bearing of FF' and the length of FA. These can now be computed and thus the missing parts are found. The computations are shown

below, where it is found that the bearing of CD is S 4°42′ W, and the length of FA is 644.25.

$$\frac{212.35}{55.44} = 3.83026 = \text{tan bearing } AF' = \text{N } 75°22' \text{ E.}$$

$$AF' = \frac{212.35}{\sin 75°22'} = \frac{212.35}{0.96756} = 219.47$$

$$\frac{\sin F}{219.47} = \frac{\sin A}{682.70}, \text{ or } \sin F = \frac{219.47 \times 0.99995}{682.70} = 0.32145$$

and $F = 18°45'$, then the bearing of FF' is $18°45' - 14°03' =$ N 4°42′ E. Hence,

$$CD = \text{S } 4°42' \text{ W, and } F' = 75°22' - 4°42' = 70°40'$$

Then

$$FA = \frac{\sin F' \times 219.47}{\sin F} = 644.25$$

Course	Bearing	Distance	Latitudes N	Latitudes S	Departures E	Departures W
A B	N 57° 13′ E	695.44	376.55		584.67	
B C	S 69° 53′ E	675.38		286.67	611.52	
C D	——	682.70				
D E	S 50° 04′ W	348.58		223.75		267.29
E F	N 75° 12′ W	741.14	189.31			716.55
F A	N 14° 03′ W	——				
			565.86	510.42	1196.19	983.84
			510.42		983.84	
			55.44		212.35	

FIG. 7-9. Calculation of Missing Parts.

7-22. Accuracy of Calculated Areas An important principle that applies to the accuracy of computed areas is that the percentage of error in an area is double the percentage of error in the perimeter. This follows from the geometrical theorem that areas of similar figures are to each other as the squares of homologous sides, and from the principle stated in Art. 7-3, that the percentage of error in the square of a number is double the percentage of error in the number itself.

This relation is readily derived also by differentiation. If A represents the area of a figure, and if x represents the side of an equiva-

lent square to which any area may be reduced, and further, if dA is the error in the area occasioned by an error dx in the perimeter, then

$$A = x^2, \text{ and } dA = 2x\,dx$$

Dividing by A,

$$\frac{dA}{A} = \frac{2x\,dx}{x^2} = 2\left(\frac{dx}{x}\right)$$

But $\frac{dA}{A}$ is the percentage of error in the area, and $\frac{dx}{x}$ is the percentage of error in the perimeter. Therefore, the percentage of error in the area is equal to two times the percentage of error in the perimeter.

It has been stated that good accuracy in taping is expressed by the ratio 1/5000. If the angles of a survey are measured with corresponding accuracy, then the total error of closure due to errors both in distance and in angles may be said to be 1/2500. Hence, according to the principle stated above, if the ratio (percentage) of error in the perimeter is 1/2500, the ratio (percentage) of error in the area will be 1/1250. This accuracy yields only four significant figures in the result.

In the example of Fig. 7-1, it may be seen that the distances are given to four significant figures only, and consequently, by Art. 7-3, the area could not have more, and it should properly be written as 561,200 sq ft or 12.88 acres.

Office Problems

Note: Observe carefully the principles of significant figures in all computed values.

7-1. A rectangular strip of concrete slab is measured and found to be 26.2 ft wide and 243.6 ft long. If it is assumed that the unknown error in each measurement is 0.1 ft, compute (a) the area as measured, (b) the percentage of error in each dimension and in the area, and (c) the error in the area in square feet. State the number of significant figures in the computed area.

Answer Area = 6382.32 sq ft

Percentage of error in width = 0.382
Percentage of error in length = 0.041
Percentage of error in area = 0.423

Error in area = 27.0 sq ft

There are three significant figures in the area; hence, to be consistent with the measurements, the area should be recorded as 6380 sq ft.

7-2. Compute the area of a triangle for which the three measured sides are $a = 1942.9$ ft, $b = 1455.7$ ft, and $c = 1423.6$ ft. Calculate by the use of logarithms and check by natural numbers.

7-3. Compute the values of the angles in the triangle of Prob. No. 7-2. Calculate by the use of logarithms and check by natural numbers.

7-4. Given the following data for a compass survey:

(a) Calculate the latitudes and departures, express the total error of closure as a ratio, and balance the survey by the compass rule.

(b) Calculate the coordinates of the vertices and compute the area.

Course	Bearing	Distance
AB	N 62°30′ E	1411
BC	S 6°45′ E	1395
CD	N 80°15′ W	1172
DA	N 26°00′ W	601

7-5. Given the following data for a transit, interior angle survey:

(a) Calculate the bearings (assuming S 79°00′ W to be the bearing of the course *AB*), the latitudes and departures and balance the survey.

(b) Calculate the coordinates and the area.

Note that the first angle given is that at point *A* and that the magnetic bearings in this problem and Prob. 7-6 serve only to provide a rough check on the calculated bearings.

Course	Distance in Feet	Interior Angle	Magnetic Bearing
AB	417.26	80°46′	S 79°00′ W
BC	219.78	132°00′	S 31°15′ W
CD	374.63	112°41′	S 36°15′ E
DE	318.25	122°48′	N 86°45′ E
EA	551.40	91°42′	N 1°30′ W

7-6. Given the following data for a transit azimuth survey:

(a) Find the bearings, from the given azimuths, compute the latitudes and departures, and balance the survey.

(b) Compute the coordinates and find the area.

Course	Distance in Feet	Azimuth	Magnetic Bearing
AB	2538.1	209°37′	S 29°30′ W
BC	3923.7	96°01′	S 83°30′ E
CD	4973.5	357°46′	N 2°00′ W
DE	3698.5	269°26′	S 89°30′ W
EA	2630.0	151°43′	S 28°15′ E

7-7. To determine the area of a tract of land bordered by the bank of a stream, offsets from a transit line were measured at regular intervals of 15 ft, with results as follows: 20.2, 17.5, 19.8, 27.3, 30.4, 24.6, 16.1, 17.0, and 22.2. Find the area by the use of Simpson's one-third rule.

7-8. Find the area of Prob. 7-7 by the use of the trapezoidal rule.

7-9. The area of an irregular tract was measured by offsets from a transit line AB, with results as follows:

Distance from Station A in Feet	Offset in Feet
0	15.0
15	21.2
40	14.5
60	24.0
90	28.6
110	8.2

Find the area by the method of coordinates.

7-10. A concrete pier is 20 ft high, 8 ft square at the bottom, and 3 ft square at the top. Find the volume (a) by the average-end-area method, and (b) by the prismoidal formula.

7-11. Use the balanced departures and latitudes of Fig. 7-1 and compute the area by the method of Double Meridian Distances.

7-12. Use the balanced departures and latitudes of Fig. 7-1 and compute the area by the method of Double Parallel Distances.

7-13. Given the data of Fig. 7-1 and assume that the length and bearing of side CD are missing. Find the length and bearing of CD.

7-14. Given the data of Fig. 7-1 and assume that the bearing of CD and the length of DE are missing. Find the missing data.

7-15. Given the data of Fig. 7-1 and assume that the length of AB and the bearing of CD are missing. Find the missing data.

CHAPTER 8

HORIZONTAL AND VERTICAL CURVES

8-1. Remarks Route surveying deals with all the field work and office studies performed in connection with the investigation of any route of transportation and the detailed layout of it. Route projects such as those for railroads and highways are so designed as to satisfy specific geometric criteria with respect to horizontal and vertical alignment.

The horizontal alignment of such projects consists of straight lines, termed *tangents*, connected by curves. The curves are usually arcs of circles or of *spirals*. The use of spiraled or *easement curves*, which provide a gradual transition between the tangents and the circular arcs, is not treated in this book.

The vertical alignment consists of straight sections of grade line connected by vertical curves. These curves are always parabolic in form because certain characteristics of the parabola facilitate the calculation and layout of the curve. The transverse section of highway pavements is likewise built to a parabolic form.

8-2. Route Surveys The transit party for route surveys usually consists of a chief of party, transitman, two tapemen, rear flagman, stakeman, and axmen as needed.

Stakes are numbered consecutively and set at 100-ft intervals along the line. These stakes mark the points to be used by the level party in taking profile levels (see Art. 3-32). The term *station* is used to designate the number of any given stake, and hence the distance from the point of beginning. Thus a driven stake numbered 74 is called *station* 74 and indicates that the point is 7400 ft from the point of beginning. The term *plus station* refers to any point inter-

mediate between two *full stations*. Thus 74 + 43.2 refers to a point 43.2 ft beyond station 74 and indicates a point 7443.2 ft from the initial or zero station. Any transit point is termed a *transit station* whether it is on a route survey or not.

A transit-station marker along a route survey is termed a *hub* and consists of a stake, preferably 2 in. × 2 in. square, driven flush with the ground and marked by a *guard stake*. The latter is a stake driven in the ground, slanting transverse to the centerline, its top being a few inches above and nearly over the hub.

Surveys for the location of route projects are usually made in two parts, namely, the *preliminary* and the *location* surveys.

A preliminary survey is made to compare the relative merits of two or more possible routes. It is made with only ordinary precision and consists of a transit traverse, a line of profile levels, and frequently a topographic survey of a strip of territory along the route. The method is that of an angle-to-right traverse as described in Art. 6-4.

A location survey is made with a higher precision than the preliminary, the curves are staked out, and curve points are referenced for permanence. This survey establishes the permanent location of the centerline, and it is from the center stakes set on this line that the slope stakes for the earthwork are located.

HORIZONTAL CURVES

8-3. Definitions and Curve Elements The various definitions and curve elements applicable to circular curves are as follows:

Degree of Curve, D. As used in this text, degree of curve (see Fig. 8-1) is defined as the central angle subtended by a 100-ft chord. This is termed the *railroad*, or *chord, definition* of degree of curve. When D is 1°, the curve is called a one-degree curve; when D is 2°, it is called a two-degree curve, etc.

Point of Curve, P.C., is the point at which the curve departs from the tangent as one proceeds around the curve in the direction in which the stationing increases (see Fig. 8-2).

Point of Tangent, P.T., opposite to the *P.C.*, marks the end of the curve and the beginning of the tangent.

Point of Intersection, P.I. The tangents to a curve, produced, meet at a point called the *Point of Intersection, P.I.*

The *Intersection Angle, I*, is the angle formed by the intersection of the two tangents at the *P.I.*

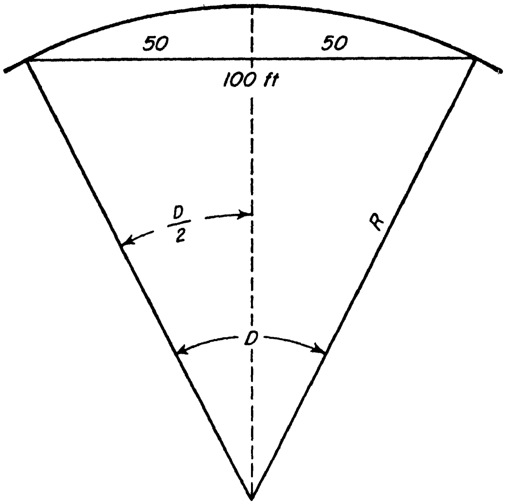


Fig. 8-1. Degree of Curve.

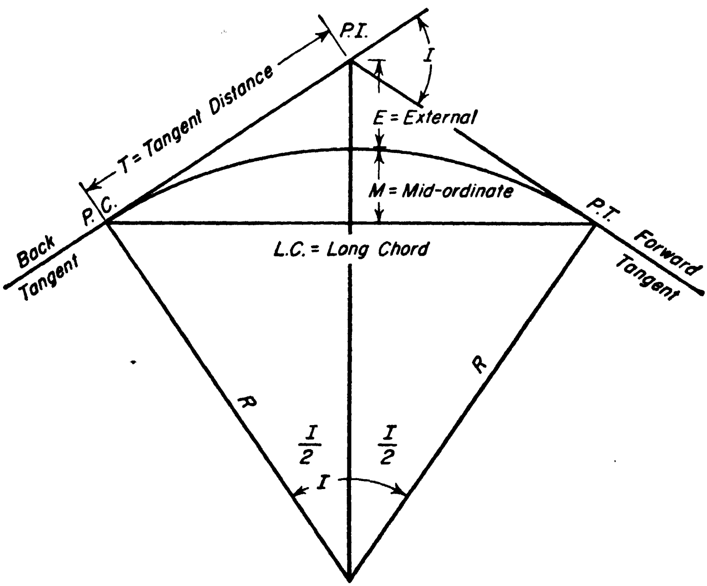


Fig. 8-2. Elements of a Circular Curve.

The *Tangent Distance, T*, is the distance along the tangent from the *P.C.* or *P.T.* to the *P.I.* From Fig. 8-2 it is evident that

$$T = R \tan \frac{I}{2} \tag{8-1}$$

The External, E, is the distance from the mid-point of the curve to the *P.I.* Evidently,

$$\frac{E + R}{R} = \sec \frac{I}{2}, \quad \text{or} \quad E = R \sec \frac{I}{2} - R$$

Then
$$E = R\left(\sec \frac{I}{2} - 1\right) \tag{8-2}$$

The Mid-Ordinate, M, is the perpendicular distance from the mid-point of the curve to the long chord.

Then
$$\frac{R - M}{R} = \cos \frac{I}{2} \quad \text{or} \quad M = R - R \cos \frac{I}{2}$$

Hence,
$$M = R\left(1 - \cos \frac{I}{2}\right) \tag{8-3}$$

The Long Chord, L.C., is the chord joining the *P.C.* and *P.T.*

$$\frac{L.C.}{2} = R \sin \frac{I}{2} \quad \text{or} \quad L.C. = 2R \sin \frac{I}{2} \tag{8-4}$$

8-4. Relation Between *R* and *D* From Fig. 8-1 it is evident that

$$R = \frac{50}{\sin D/2} \tag{8-5}$$

If D is 1°, R becomes 5729.65 ft. Also, since R varies inversely as $\sin D/2$, R will vary inversely (and almost exactly so) as D for the small values of D associated with the relatively flat curves of modern route engineering practice.

In general, for approximate calculations

$$R = \frac{5730}{D} \tag{8-6}$$

or
$$D = \frac{5730}{R} \tag{8-7}$$

Equation (8-7) permits the degree of curve to be readily obtained if the curve is defined by its radius. It is to be noted that a sharp

curve has a short radius and a flat curve a long radius. The degree of curve on modern, high-speed highways is usually less than 4°.

It is seldom necessary to calculate R from Eq. (8-5). When D is given and the curve elements involving R are to be calculated, the value of R should be taken from Table VIII.

8-5. Length of Curve The length of curve, L, is the sum of the chord distances around the curve from the $P.C.$ to the $P.T.$ This will invariably involve subchord lengths (see Fig. 8-3) adjacent to the $P.C.$ and $P.T.$ as well the 100-ft chord lengths between full stations on the curve.

From the definition of D it follows that

$$L = \frac{I}{D} \times 100 \tag{8-8}$$

8-6. Tables of T and E It is to be noticed from an examination of Eqs. (8-1) and (8-2) that T and E, respectively, vary directly as R. This means they vary inversely (and almost exactly so) as D. Accordingly, values of T and E for any degree of curve, for a particular value of I, can be easily obtained by dividing the values of T and E for a 1° curve, having the same value of I, by the value of D. Hence,

$$T = \frac{T_{1^\circ}}{D} \tag{8-9}$$

and

$$E = \frac{E_{1^\circ}}{D} \tag{8-10}$$

Values of T_{1° and E_{1° are tabulated in Table IX for various values of I. For flat curves the agreement in the values of T calculated from Eqs. (8-1) and (8-9) should be very close. The value obtained from Eq. (8-1) is exact.

8-7. Principle of Deflection Angles The deflection-angle method is employed almost exclusively in laying out circular curves. It is based on the geometric principle that the angle between a tangent and a chord at a point on a circle is equal to one-half the angle subtended by the chord. Fig. 8-3 depicts the essential relationships between deflection angles at the $P.C.$ and the corresponding central angles. The first subchord is denoted by c and the first deflection angle, $(d/2)$, is calculated by

$$\left(\frac{d}{2}\right) = \frac{c}{100} \times \frac{D}{2} \tag{8-11}$$

The increment of deflection for a full 100-ft chord is $D/2$. The last increment of deflection (for the final subchord) is likewise calculated from Eq. (8-11).

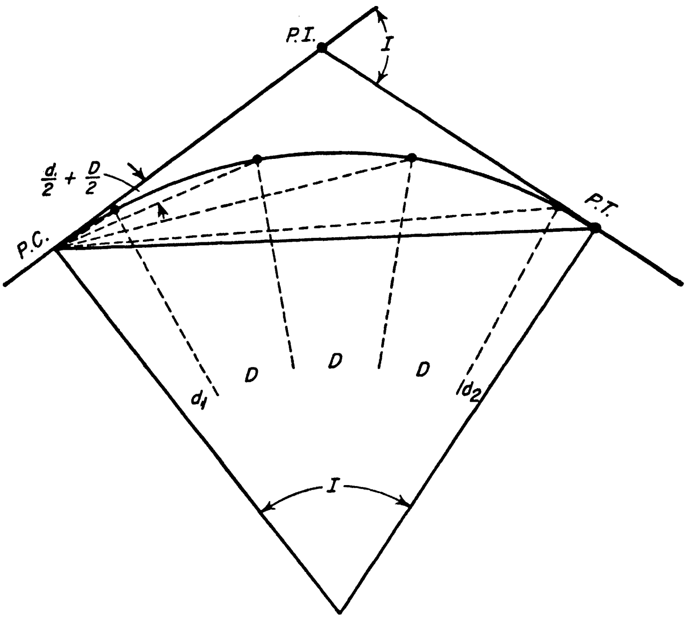


FIG. 8-3. Principle of Deflection Angles.

8-8. Staking Out a Curve Before a curve can be staked out, it is necessary to extend the two established tangents to an intersection at the *P.I.*, measure *I*, and select the value of *D*. Sometimes the value of *D* is fixed by topographic or other considerations. It is important to observe that the station of the *P.I.* is determined by continuing the stationing along the *back tangent* to that point, and that the station of the *P.T.* (along the *forward tangent*) is ascertained by adding the calculated length of curve to the stationing of the *P.C.* From these data the necessary computations are made, after which the field work is executed. The successive steps in this procedure are as follows:

1. The various functions of the curve are computed by the use of

formulas of Art. 8-3 or by the use of suitable tables. See Tables VIII and IX.

2. The deflection angles are computed and properly arranged in the field notebook.

3. The distance T is measured from the $P.I.$ along each of the tangents to set the $P.T.$ and the $P.C.$

4. The transit is set up at the $P.C.$ and properly oriented.

5. The deflection angles are turned off with the transit and corresponding chords are measured, thus to establish the successive points along the curve.

The above procedure will be illustrated by the use of an example.

Given the following data for a circular curve: $I = 26°40'$; $D = 4°00'$; $P.I.$ = station 45 + 59.5. Required: the data and description of the field procedure to stake out this curve.

1. The various functions are found by the use of the formulas given above and T and E are checked with Table IX.

$$R = 1432.7 \text{ ft} \qquad \text{Table VIII}$$

$$T = R \tan \frac{I}{2} = 339.5 \text{ ft}$$

also, $$T = \frac{1358.0}{4} = 339.5 \text{ ft} \qquad \text{Table IX}$$

$$E = R \sec \frac{I}{2} - R = 39.7 \text{ ft}$$

also, $$E = \frac{158.7}{4} = 39.7 \text{ ft} \qquad \text{Table IX}$$

$$M = R\left(1 - \cos \frac{I}{2}\right) = 38.6 \text{ ft}$$

$$L.C. = 2R \sin \frac{I}{2} = 660.7 \text{ ft}$$

$$L = \frac{I}{D} \times 100 = 666.7 \text{ ft}$$

2. It is customary to arrange the notes for transit route surveys from the bottom of the page upward. This arrangement permits field sketches to be entered on the right-hand page of the notebook in a natural relation to the forward direction of the centerline of the survey. Thus a feature which appears on the right of the survey center-

line may be sketched on the right of the notebook center line. Following this procedure and the principles stated above, the deflection angles are calculated and arranged as shown in Fig. 8-4.

There it will be seen that the station number of the *P.C.* is found

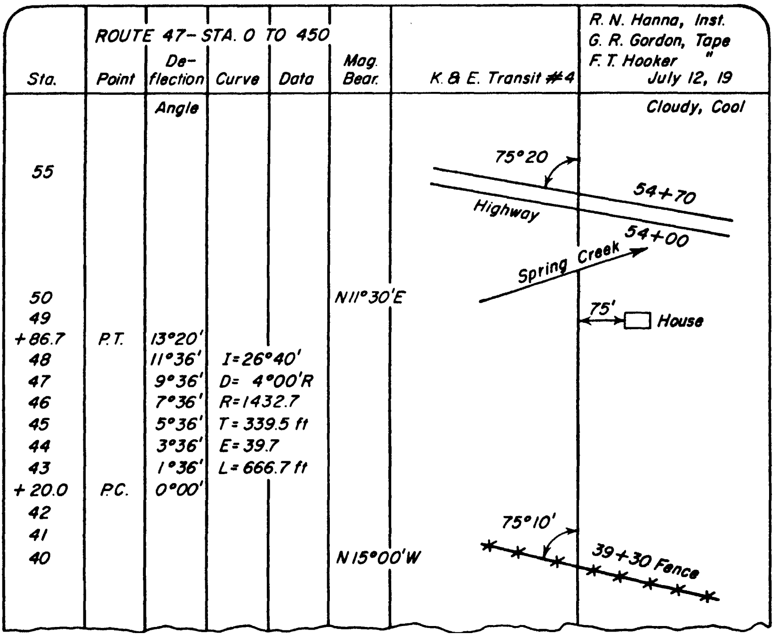


FIG. 8-4. Notes for a Curve.

by subtracting the tangent distance from the *P.I.*, after which the forward stationing proceeds along the curve to the *P.T.* Thus:

$$
\begin{array}{rcl}
45 + 59.5 & = & P.I. \\
-3 + 39.5 & = & T \\
\hline
42 + 20.0 & = & P.C. \\
+6 + 66.7 & = & L \\
\hline
48 + 86.7 & = & P.T.
\end{array}
$$

Since the station of the *P.C.* is 42 + 20.0, the distance from the *P.C.* to the first station on the curve (i.e., station 43 + 00) is 80.0 ft. Accordingly, the deflection angle at the *P.C.* for a point on the curve 80 ft distant will be $0.8D/2 = 1°36'$. The next deflection angle for station 44 will be $1.8D/2 = 1°36' + 2°00' = 3°36'$; etc. The deflec-

tion angle for the *P.T.* will equal the deflection angle for station 48 plus that for a distance of 86.7 ft, or

$$11°36' + \frac{0.867D}{2} = 11°36' + 1°44' = 13°20'$$

Since this value is seen to be equal to $I/2$, a check is thus provided on the computation of all deflection angles.

3. The tapemen measure the distance T along the forward tangent from the *P.I.*, and set the *P.T.*, marked with its proper station number 48 + 86.7. Then they measure the same distance T along the initial tangent from the *P.I.* and set the *P.C.*, being station 42 + 20.0.

4. The transitman then sets his instrument up at the *P.C.*, sets the vernier A to read zero, and sights the *P.I.* with the telescope normal.

5. The transitman now turns the first deflection angle, 1°36′ on the A vernier and the tapemen measure a distance of 80.0 ft from the *P.C.*, thus to locate the first station, 43 + 00, on the curve. Next the transitman turns off the deflection angle for station 44 + 00 as shown in the notes, 3°36′, and the tapemen measure a full 100-ft station from the stake previously set, to locate station 44 + 00. In like manner the successive stations are located around the curve. When the *P.T.* is reached and located, as were the previous stations, a check is provided by its proximity to the point previously set as described under step 3 above.

If all the computations and field work were without error, the two points would coincide; but this rarely happens, and, if the distance between the points is not more than $L/20$ ft (i.e., the length of the curve in stations, divided by 20) the results, for ordinary conditions, may be considered to be satisfactory.

8-9. Intermediate Setup on a Curve Because of some obstruction, or the great length of the curve, it is frequently necessary to set the instrument at a station on the curve. In this case, the manner of orienting the transit and of turning off the subsequent deflection angles will now be described. The situation is illustrated in Fig. 8-5, and it is supposed that the notes have been computed and arranged similar to those in Fig. 8-4. The *P.C.* is at station 0 + 00.

It is evident in the figure that station 3 and those following are not visible from the *P.C.* Hence, it is desirable to set the instrument at station 2 and to continue setting stations on the curve from that

point. Accordingly, stations 1 and 2 are set as usual from the *P.C.* and the instrument is then removed to station 2.

From the figure, angle $\alpha_1 = \alpha_2 = \alpha_3$; also $\beta_1 = \beta_2$. Hence $\alpha_3 + \beta_2 = \alpha_1 + \beta_1$. But the angle $\alpha_1 + \beta_1$ is equal to the deflection angle of

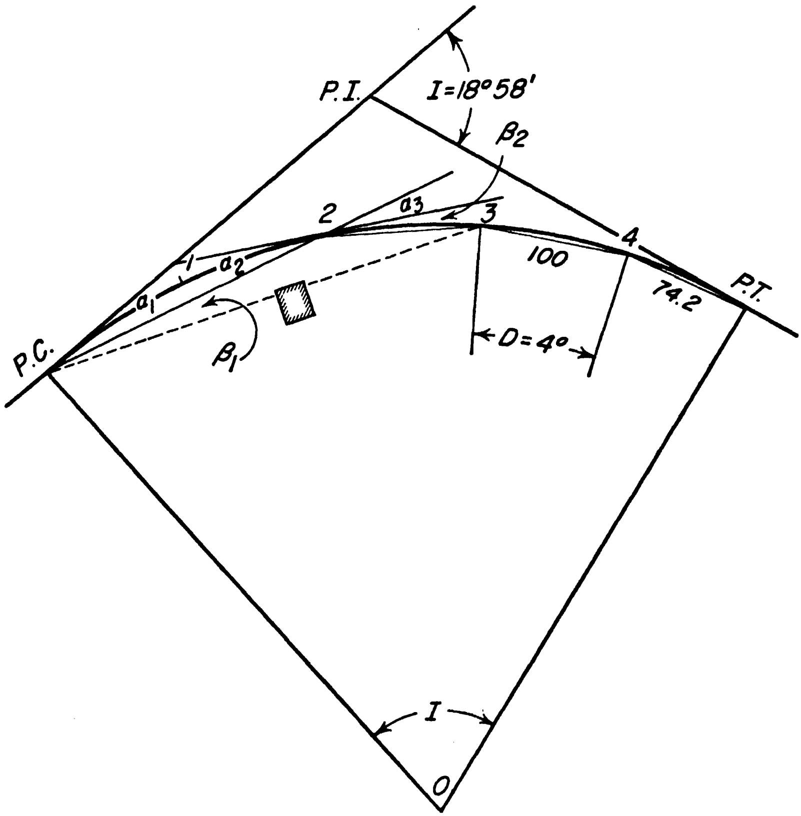


FIG. 8-5. Intermediate Setup on a Curve.

station 3 from the *P.C.* Hence, if the line of sight is directed along the chord 2 − *P.C.* produced, with the vernier reading zero, obviously the correct deflection angle to set station 3 will be $\alpha_3 + \beta_2$, the same angle which would have been used if this station could have been set from the *P.C.*

This procedure may be stated as follows: with the transit at station 2, set the *A* vernier to read zero, and with the telescope inverted, sight the *P.C.* Return the telescope to the normal position and turn

the deflection angle for station 3 which has previously been calculated and recorded in the notes.

For example, if this is supposed to be a 4° curve, then at station 2, the vernier would be set at 0° for the backsight on the *P.C.*, the telescope inverted and the vernier set at 6°00′ to locate station 3.

Again, if it were necessary to set the transit at station 4, it would be found by similar analysis that the correct procedure would be to set the vernier to read the deflection angle previously computed for the backsight, i.e., for station 2, sight on station 2 with the telescope inverted, reinvert the telescope and turn off the deflection angle which has previously been recorded in the notes for each following station.

For example, with the transit at station 4, a backsight would be taken on station 2 with the telescope inverted and the vernier reading 4°. Then the telescope would be returned to the normal position and the angle of $9°29' = I/2$ would be turned off to locate the *P.T.*

The procedure described above can be summarized by the following general rule: *With the transit in position at any station on a curve, the backsight is taken with the telescope inverted and the A vernier set to read the deflection angle of the point sighted. The telescope is then inverted and the following stations on the curve are located by using the deflection angles previously computed and recorded in the notebook.*

VERTICAL CURVES

8-10. Mathematical Principles Three mathematical properties of the parabola render it especially convenient to use as a vertical curve to connect two intersecting grades. These properties are illustrated in Fig. 8-6 and may be stated as follows:

1. That portion of the axis shown as AV is bisected by the curve at B.

2. Offsets from a tangent to the curve vary as the square of the distance from the point of tangency.

3. For a parabola used as a vertical curve, the second differences of the elevations of points spaced at equal horizontal intervals along the curve are equal.

The applications of these principles may be indicated as follows:

(a) The distance $AB = BV$.

(b) By the method given in this article, the offset, O_3, at the vertex is found directly from the given data. Then if the distances from

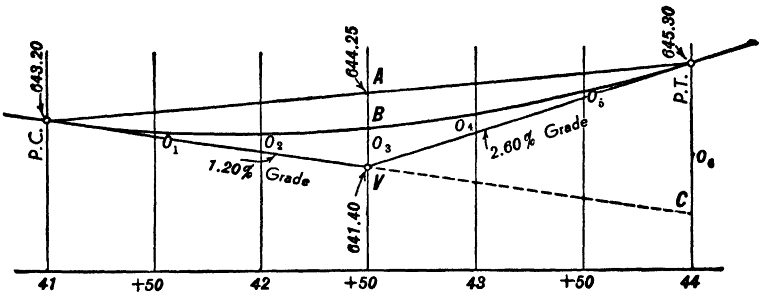
the point of tangency, *P.C.*, are expressed in stations, we have the relations

$$\frac{O_2}{O_3} = \frac{1.0^2}{1.5^2} \text{ or } O_2 = \frac{4}{9}O_3; \text{ also } \frac{O_1}{O_3} = \frac{0.5^2}{1.5^2} \text{ or } O_1 = \frac{1}{9}O_3;$$

$$\text{also } \frac{O_6}{O_3} = \frac{3.0^2}{1.5^2} \text{ or } O_6 = 4 \times O_3.$$

The offsets O_4 and O_5 are equal to O_2 and O_1, respectively.

(c) The second differences between the curve elevations are given in the table and are seen to be constant except for the small variations due to dropping thousandths in the computed elevations.



Sta.	Grade Elev.	Offsets	Curve Elev.	1st Diff.	2nd Diff.
41 P.C.	643.20	0.00	643.20		
				+0.44	
+50	642.60	.158	642.76		+0.31
				+0.13	
42	642.00	.632	642.63		+0.32
				−0.19	
+50	641.40	1.425	642.82		+0.32
				−0.51	
43	642.70	.632	643.33		+0.32
				−0.83	
+50	644.00	.158	644.16		+0.31
				−1.14	
44 P.T.	645.30	0.00	645.30		

FIG. 8-6. Vertical Curve.

8-11. Computations of a Vertical Curve The known data for the computation of a vertical curve include the station and elevation of the vertex, and the gradients of the two intersecting grade lines. Fig. 3-23*b* shows a plotted profile which usually provides the basis for determining the gradients. The engineer selects the length of curve as some whole number of 100-ft stations. The length of the vertical curve is the horizontal distance from *P.C.* to *P.T.* Sometimes the length of curve is fixed by certain design criteria such as the minimum sight distance over a summit vertical curve.

It is worthwhile emphasizing here that the term *grade* has two different meanings in engineering practice. It can indicate the slope, or gradient, of a line. Hence, a 1% grade is one which rises or falls 1 ft per 100 ft of horizontal distance. The term grade also indicates the final or finish elevation of some part of an engineering project.

Different methods are available for computing and checking the elevations for a vertical curve, but the one given here is simple and convenient. An example will illustrate the method (see Fig. 8-6).

Given: a −1.20% grade intersects a +2.60% grade at station 42 + 50; elevation 641.40. Required: Connect these grades with a vertical curve 300 ft long, using 50-ft stations.

First, the elevations of the 50-ft stations along the original grade lines are calculated.

Next, the value of the offset from the vertex to the curve is found as being one half the distance from the vertex to the chord connecting the $P.C.$ and $P.T.$ Elev. of $A = \dfrac{\text{elev. of } P.C. + \text{elev. of } P.T.}{2} =$ 644.25. Then $AV = 644.25 - 641.40 = 2.85$ ft; and $VB = 1.425$. This value, VB, may be taken as the offset from the tangent at V.

From Fig. 8-6 it is evident that the distances along the tangent from the $P.C.$ to stations 41 + 50 and 42 are 1/3 and 2/3, respectively, of the distance from $P.C.$ to V. Since the offset O_3 at V is 1.425 ft, then, according to one of the principles stated above,

$$\text{the offset } O_1 = \left(\frac{1}{3}\right)^2 \text{ of } 1.425 = 0.158 \text{ ft}$$

$$\text{and the offset } O_2 = \left(\frac{2}{3}\right)^2 \text{ of } 1.425 = 0.632 \text{ ft}$$

As stated previously, offsets O_5 and O_4 are equal to O_1 and O_2, respectively.

Having computed the offsets, the elevations of the corresponding points on the curve are readily found, and the results are conveniently arranged as shown.

According to another principle of Art 8-10, the second differences of the curve elevations should be equal, and this condition is found to be true. This condition affords an excellent check on all the computations but only when the points are at equal distances along the curve.

In all vertical-curve computations due regard must be taken of algebraic signs.

As to significant figures, it is usually satisfactory if the curve elevations are correct to the nearest one hundredth of a foot. This requires that gradients be found to the nearest one hundredth of a per cent and that offsets be figured to the nearest one thousandth of a foot.

Slight variations may be found in the second differences due to figures dropped in finding the elevations, but such variations should not be greater than one or two hundredths of a foot.

8-12. Pavement Crowns The parabola is used in the design of crowned pavements. Knowing the total rise or crown at the center of a pavement and the width, it is a simple matter to find the elevation of any intermediate point along a cross section of the roadway.

Thus in Fig. 8-7 the crown is represented by O_2, the rise, or height

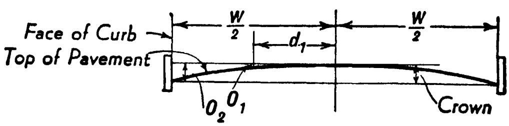


FIG. 8-7. Pavement Crown.

of the center of the pavement above the gutter elevation. The offset O_1, at any other distance d_1, is given by the relation

$$\frac{O_1}{O_2} = \frac{d_1^2}{(W/2)^2}$$

For example, a pavement has the following dimensions: $W = 28$ ft, $d_1 = 7$ ft, and $O_2 = 4$ in. Then $O_1 = \frac{1}{4} \times O_2 = 1$ in.

Office Problems. The solution of the following problems, relating to circular curves, should include the computation of the deflection angles to stake out the curves and the arangement of the data in good notebook form. Use five places of the tables and calculate all values to five significant figures.

8-1. Given: $I = 32°10'$; $D = 4°00'$; *P.C.* = 42 + 50.0. Find *R, T, E, M*, and *L*, using the formulas; and check *R, T*, and *E* from Tables VIII and IX.

8-2. Given: $I = 28°30'$; $T = 350.0$ ft; *P.I.* = 74 + 40.0. Find *D, R*, and *L*. Calculate *D* to the nearest 0.1′.

8-3. Change the value of *D* of Prob. 8-2 to the nearest even 10′, and calculate the corresponding values of *R, T, L*, and *E*.

8-4. Given: $I = 24°20'$; $R = 1500.0$ ft; $P.C. = 30 + 25.0$. Find D, T, E, and L.

8-5. Given: $I = 34°42'$; $E = 70.00$ ft; $P.I. = 125 + 50$. Find R, D, T, and L.

8-6. A +2.40% grade meets a −1.80% grade at station 42 + 50, elevation 746.40. Connect these grades with a vertical curve 300 ft long, using 50-ft stations.

8-7. A −0.40% grade meets a +2.7% grade at station 18 + 00, elevation 974.60. Connect these grades with a vertical curve 200 ft long, using 25-ft stations.

8-8. The elevation of the top of pavement on the centerline at station 20 + 00 is 645.30. The grade is −0.80%; the pavement is 32 ft wide face to face of curb; and the crown is 4 in. What will be the grade elevation of a point at station 20 + 40 and 12 ft distant, at right angles, from the centerline?

Field Problem 8-1. Staking Out a Curve, Setting All Curve Points From the *P.C.*

Procedure.—Use the data of Office Prob. 8-1. These data show the calculated values of the functions of the curve and the deflection angles to each station on the curve. Choose a suitable point for the *P.I.* and set up the transit. Assume a direction for the forward tangent and tape the distance T and set the *P.T.* Invert the telescope, measure the intersection angle I, and measure the distance T and set the *P.C.* = station 42 + 50.00. Set the transit at the *P.C.* and with the vernier at 0°00′ sight the *P.I.* Turn off the deflection angle and measure the distance of 50 ft to set station 43 + 00 on the curve. Continue to turn the deflection angles and measure the 100-ft distances to set the remaining points on the curve. The last distance and deflection angle should fall on the *P.T.* previously set, but will not coincide with it because of the errors of measurement. The total error of closure should not exceed 0.5 ft.

It is a good check, also, to calculate the station and deflection angle to the mid-point on the curve. When this point is set in the field, the external distance is measured and should check the computed value within 0.3 ft.

The transit stations consist of a hub driven flush with the ground on which a tack is set to mark the point. A guard stake is driven beside the hub, marked with the station number.

The checks should be clearly shown in the notebook, and a paragraph should explain the procedure followed. Follow the form shown in Fig. 8-4, except that bearings need not be used.

Field Problem 8-2. Staking Out a Curve, Setting the Transit at Two Points on the Curve.

Procedure.—Use the data of Office Prob. 8-3 and proceed as in Field Prob. 8-1 above until the *P.C.* is fixed and station 73 is

reached. This point should be carefully set and tacked for an instrument station. The transit is then moved to station 73, oriented by the method explained in Art. 8-9, and the curve continued to the mid-point, which is set and the external measured as a check. The calculated and measured values of E should agree within 0.3 ft. Continue the curve to station 76, set a transit station, move the transit to this point, backsight to station 73, and continue the curve to the *P.T*. The final error of closure should not exceed 0.4 ft. The checks and procedure should be clearly shown in the notes.

CHAPTER 9

STADIA SURVEYING

9-1. Remarks The stadia method of measuring distances is very rapid and convenient. The accuracy attained is such that under favorable conditions the error will not exceed 1/1000, and if the purpose of a survey does not require greater accuracy, the method is unexcelled. Also, on surveys of higher accuracy it provides a ready check on distances measured with the tape. In combination with the measurement of vertical angles, the method is similar to trigonometric leveling. The stadia method is especially well adapted to mapping requirements and is almost universally used for locating the details and contour points of topographic surveys.

9-2. Instruments and Method The instrumental equipment for stadia measurements includes a graduated rod and an engineer's transit the telescope of which is provided with two additional stadia cross-wires. These are mounted on the cross-wire ring, one above and the other below the middle horizontal cross-wire. The rod divisions are usually in units of feet and tenths of feet and arranged in various patterns, one of which is shown in Fig. 9-1. Of the many patterns of rods that have been designed, probably none is any better for general use than that shown here. For short sights, up to about 500 ft, the ordinary leveling rod may be used; beyond that distance the fine markings of a level rod become indistinct, and a pattern of larger divisions is necessary.

A reading is taken by setting the lower cross-wire on a full foot mark on the rod and noting where the upper cross-wire cuts the rod. The difference between the two readings is called the *interval,* or *intercept,* and is a measure of the distance from the instrument to the rod.

9-3. The Stadia Theory for Horizontal Sights If a treatment of the optics of the telescope is omitted, the theory of the stadia is simple. In Fig. 9-2 is illustrated a telescope, the plumb line, and a rod intercept, r. Two rays of vision are drawn, namely those emanating from the cross-wires and parallel with the optical axis. These rays are refracted by the objective lens and pass through a focal point at a distance F in front of the lens, and intersect the rod as shown.

If the distance between the cross-wires is represented by i, then in the two similar triangles we have the relation

$$\frac{d}{r} = \frac{F}{i}$$

from which $d = (F/i)r$, and from the figure, the total distance from the rod to the plumb line is

$$D = \left(\frac{F}{i}\right)r + (F + c) \qquad (9\text{-}1)$$

The Constant $(F + c)$.—The distance F is the focal length of the lens and is a constant. Also, the quantity $(F + c)$ is practically a constant, since the distance c varies but a small and negligible amount when the telescope is focused on different objects. The value of this constant varies somewhat for different instruments, but the range is between about 0.8 ft and 1.2 ft, or practically 1.0 ft. When the errors in stadia measurement are considered below, it is found that the estimated error of determining distances by this method is, for average sights, ± 1 ft or more. Furthermore, it is inconsistent to consider this constant unless the systematic errors of differential refraction are likewise considered. These sources of error could not be overlooked on careful surveys of wide extent, but for ordinary construction surveys and for topographic mapping where an accuracy between 1/500 and 1/1000 is sufficient, the quantity $(F + c)$ may properly be neglected.

It may be added that many instruments of recent design have interior focusing telescopes; and for such telescopes the quantity $(F + c)$ can be disregarded under all conditions.

The Constant (F/i) *or* k.—Since the principal focal

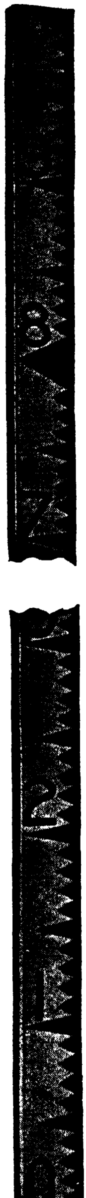


Fig. 9-1. Rod for Stadia Surveying.

length F and the distance i between cross-wires are constants for any telescope, the ratio (F/i) also is a constant and is represented by k. Then, neglecting $(F + c)$, Eq. (9-1) becomes $D = kr$. The quantity k is the factor by which each rod intercept, r, is multiplied

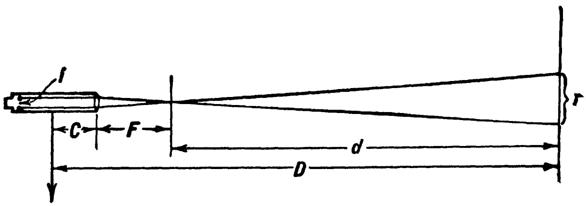


FIG. 9-2. Stadia Principle for Horizontal Sights.

to determine the distance D, and, for convenience, it is made to equal 100 as nearly as can be. It may be noted that it is of no importance to know the value of either F or i separately.

Since all rod intervals are to be multiplied by this factor k, it is important that its value be determined with some precision. This may be accomplished conveniently as follows: set a series of pins by pacing at distances of about 100, 300, 500, and 700 ft from the instrument. Read the intercept carefully at each pin, estimating all readings to 0.01 ft. Finally, measure the distance to each pin with the tape. The ground should be nearly level, but the telescope bubble need not be centered. The bottom cross-wire should be set on a full foot mark to facilitate the intercept reading. It will increase the accuracy of the rod readings if the field conditions are such that the sun's rays fall on the face of the rod. Since considerable precision is necessary for this determination, $(F + c)$ may be included for an external focusing telescope and taken as 1 ft and Eq. (9-1) applied. From these data, four values of k will be found, the mean of which may be accepted for use.

It should be added that, since the intercept r depends on the rod graduations, it is desirable that the accuracy of the divisions on the stadia rod be checked by a steel tape. This is especially important if the rods in use are "hand made," i.e., constructed by the organization using them. Such rods are likely to have their lengths appreciably affected by moisture and temperature changes.

9-4. Inclined Sights On sloping ground the difference in elevation and the horizontal distance between two points can be found

by the stadia method, if, in addition to the rod interval, the angle of inclination of the line of sight is read on the vertical arc. This application of the method permits transit-stadia traversing over hilly ground, or stadia differential leveling, or a combination of both. In the use of the engineer's level, the maximum difference in elevation that can be measured at a given setup is limited by the length of the level rod. Stadia leveling, however, is not limited in this manner and hence in hilly country is far more rapid than the usual method.

In Fig. 9-3, let it be supposed that the transit is in position at station A and that the rod is held vertically at station B; that the

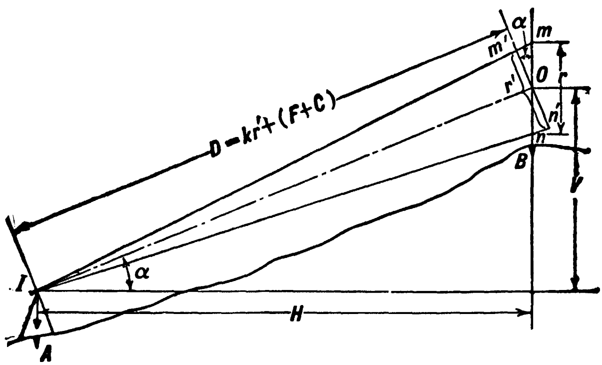


FIG. 9-3. Stadia Principle for Inclined Sights.

rod intercept is $mn = r$, and that the inclination of the line of sight as measured on the vertical arc is α; also that $m'n' = r'$ is the intercept which would be read on the rod if it were held perpendicular to the line of sight.

Since the imaginary line $m'n'$ is perpendicular to line of sight IO, angle $Om'm = On'n$ is not exactly a right angle. However, in the following demonstration it will be so considered, and the error thus introduced is negligible. With this approximation understood, the following equations may be written:

$$m'n' = mn \cos \alpha; \text{ or } r' = r \cos \alpha \tag{9-2}$$

$$D = kr' + (F + c) \tag{9-3}$$

Substituting (9-2) in (9-3),

$$D = kr\cos\alpha + (F + c) \tag{9-4}$$

The difference in elevation between I and O is given by the vertical projection V of the slope distance D.

Hence,

$$V = D\sin\alpha \quad \text{or} \quad V = kr\sin\alpha\cos\alpha + (F + c)\sin\alpha$$

from which,

$$V = kr\,\tfrac{1}{2}\sin 2\alpha + (F + c)\sin\alpha \tag{9-5}$$

Also, the horizontal distance H is equal to the horizontal projection of the slope distance D, and

$$\begin{aligned} H &= D\cos\alpha \\ &= kr\cos^2\alpha + (F + c)\cos\alpha \end{aligned} \tag{9-6}$$

From the figure it is evident that the difference in elevation between I and O, which equals V, is the same as the difference in elevation between ground points A and B, if distance OB is taken equal to IA; also, horizontal distance H is equal to the horizontal distance between points A and B. Thus, Eqs. (9-5) and (9-6) permit the determination of the difference in elevation and of the horizontal distance between any two given stations.

For reasons stated above and for usual conditions, sufficiently satisfactory results will be secured if the constant $(F + c)$ is disregarded. Hence, Eqs. (9-5) and (9-6) become

$$V = kr\,\tfrac{1}{2}\sin 2\alpha \tag{9-7}$$

and

$$H = kr\cos^2\alpha \tag{9-8}$$

Table VII has been computed by the use of these formulas, and this table can be used for all ordinary surveys.

9-5. Reduction of Stadia Notes *Tables.*—To simplify the use of Eqs. (9-7) and (9-8) of the preceding article, Table VII has been computed. This table gives the values of H and V for various vertical angles α, and for the value $kr = 100$. For other values of kr, a direct proportion is to be used.

EXAMPLE 1. Given $r = 3.17$, $k = 100$, and $\alpha = 4°20'$. Find H and V.

In Table VII, the values of H and V for $\alpha = 4°20'$ and $kr = 100$ are 99.43 and 7.53, respectively. Hence, for $kr = 317$,

$$H = 3.17 \times 99.43 = 315$$
$$V = 3.17 \times 7.53 = 23.9$$

EXAMPLE 2. Given $r = 6.37$, $k = 101.2$ and $\alpha = 3°40'$. Find H and V.

$$kr = 6.37 \times 101.2 = 645$$

Then from the table, $H = 99.59$ and $V = 6.38$ for $kr = 100$. Accordingly,

$$H = 6.45 \times 99.59 = 642$$
$$V = 6.45 \times 6.38 = 41.2$$

For the unusual cases of high accuracy, when Eqs. (9-5) and (9-6) should apply, they can be transformed into Eqs. (9-7) and (9-8) by the relations

$$\left.\begin{aligned}(F + c)\sin\alpha &= 1\text{ ft }\tfrac{1}{2}\sin 2\alpha\\ \text{and }(F + c)\cos\alpha &= 1\text{ ft }\cos^2\alpha\end{aligned}\right\}\text{very nearly}$$

Accordingly, Eqs. (9-5) and (9-6) become

$$V = (kr + 1)\,\tfrac{1}{2}\sin 2\alpha \tag{9-9}$$

$$\text{and } H = (kr + 1)\cos^2\alpha \tag{9-10}$$

from which it is evident that Eqs. (9-7) and (9-8) and Table VII may be used for this case if 1 ft is added to the value of kr before making the computations indicated in the examples given above.

It may be noted that the stadia readings contain three significant figures only; hence, the ordinary slide rule may be used in these computations.

Stadia Slide Rules.—To avoid the use of tables and further to simplify the reductions of stadia notes, special slide rules have been designed. These need not be described in detail here, but they are most convenient and should be available if many transit-stadia notes are to be reduced. This remark applies in particular to plane-table surveys where the reductions of notes are made in the field.

9-6. Transit-Stadia Traversing The principal characteristic of a transit-stadia traverse is that the distances are measured by the stadia method instead of a tape. If the ground is level, no vertical angles are required, but, as will be seen later, where the ground slope

is as much as 3°, i.e., about 5%, the vertical angle should be measured to determine the correct horizontal distance.

Permissible errors in centering the instrument over a point and in sighting an object are correspondingly larger than for a transit-tape traverse.

The form of notes for a transit-stadia traverse is illustrated in Fig. 9-4.

TRANSIT STADIA TRAVERSE — Locker 15 — 27

G. W. Chinn ⊼
H. J. Benson, Rod
May 22, 19

Sta. to Sta.		Azimuth	Rod	Vert. ∠	Hor. Dist.	Mag. Bear.	Calc. Bear.
A	F	110°52'	6.91				
A	B	231°30'	5.52	+4°26'	548	S 51°30'W	S 51°30'W
B	A	51°30'	5.50				
	C	185°35'	8.41		842	S 5°30'W	S 5°35'W
C	B	5°35'	8.43				
	D	65°45'	4.43	−3°48'	441	N 66°00'E	N 65°45'E
D	C	245°45'	4.43				
	E	83°40'	6.40		641	N 83°30'E	N 83°40'E
E	D	263°40'	6.42				
	F	10°05'	6.92		693	N 10°15'E	N 10°05'E
F	E	190°05'	6.94				
	A	290°40'	6.91		691	N 69°15'W	N 69°20'W
		110°52'					
		179°48'	=12' Error				

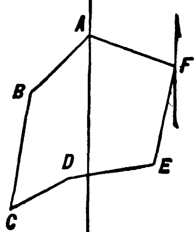


FIG. 9-4. Field Notes for Transit Stadia Traverse.

The checks to be applied are those used in other work as described in Art. 6-7.

9-7. Stadia Leveling It has been shown in a previous article how the difference in elevation between two points can be determined by a stadia reading and a vertical angle. By applying this method to a succession of stations, a line of stadia levels may thus be run. A difference between this method and that of the engineer's level is that usually, for stadia leveling, it is required to determine the elevation of the ground at the instrument station.

The instrument is set up at a convenient distance from a benchmark and the stadia interval is read by placing the bottom cross-wire on a full footmark and noting where the upper cross-wire cuts the rod. The middle cross-wire is then sighted at a point on the rod whose height above the ground is the same as the height of the

horizontal axis of the instrument above the ground. This distance is called the *height of instrument*, H.I., although this term does not have the same meaning as that used in differential leveling. The vertical angle is then read, and from these data the elevation of the transit station can be determined. The rodman then goes ahead to a turning point, and a similar series of readings is taken to determine its elevation.

Sometimes, because of obstructions, it is not possible to sight the middle cross-wire at the H.I. on the rod. If then some other point is sighted, a corresponding correction is made to the computed difference in elevation.

EXAMPLE 1. Given: Elevation of B.M. = 500.00 ft; H.I. = 4.2 ft; rod intercept = 5.13; vertical angle = $+4°14'$. Find the elevation of the instrument station.

$$\begin{aligned} \text{Difference in elevation} &= -5.13 \times 7.36 = -37.8 \\ \text{Elevation of instrument station} &= 500.00 - 37.8 \\ &= 462.2 \text{ ft} \end{aligned}$$

EXAMPLE 2. Given: Elevation of B.M. = 500.00 ft; H.I. = 4.2; $k = 102.0$; rod intercept = 6.34; vertical angle = $-5°20'$; reading taken on the 8.0 ft mark on the rod. Find the elevation of the instrument station.

$$\text{Difference in elevation} = +\left(\frac{102.0}{100} \times 6.34 \times 9.25\right) + 8.0 - 4.2 = 63.6 \text{ ft}$$

$$\text{Elevation of instrument station} = 500.00 + 63.6 = 563.6 \text{ ft}$$

In case it is of no importance to know the elevation of the instrument station, it is immaterial what point (H.I.) on the rod is sighted in measuring the vertical angle, provided the same point (H.I.) is sighted both on a backsight and the corresponding foresight.

Of course, if the ground is level so that a direct level reading can be taken on the rod, no vertical angle is required, and the differences in elevation are computed as in ordinary differential leveling.

9-8. Locating Details It is assumed that before details are located, the transit stations have been located by a traverse, and that the elevations of these stations have been determined. A discussion of the proper methods to use for various field conditions is given in Chapter 13.

For the usual topographic survey, the location of details includes the determination of the azimuth, distance, and elevation of a sufficient number of points properly to portray the significant features of the area. The elevations of some points may not be required, and for such, of course, vertical angles are not read.

The procedure may be described by referring to Fig. 9-5.

LOCATING DETAILS LAKE COMO RESERVOIR						Gurley Transit No. 7	G. B. Righter, ⊼ H. R. Linder, Notes Dan Cohen, Rod 43
Inst. at C, Elev. 633.3			H. I.=4.4			July 12, 19	
Object	Az.	Rod	Vert. Angle	Hor. Dist.	Diff. in Elev.	Elev.	
B	5°35'						
1	208°40'	2.22	-2°21'	222	-9.1	624.2	
2	258°30'	1.10	(3.9)	110	+0.5	633.8	
3	333°40'	2.43	-1°14'	243	-5.2	628.1	
4	347°15'	4.98	-2°23'	497	-20.7	612.6	
5	8°25'	4.32	(on 8.0) -3°05'	431	-26.8	606.5	
6	14°15'	3.63	-3°01'	362	-19.1	614.2	
			etc.				
B	5°37'	check	↓				

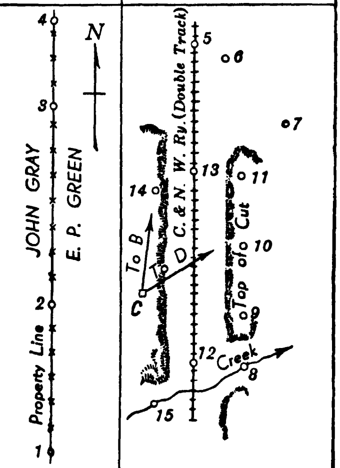


FIG. 9-5. Field Notes for Stadia Surveying.

The instrument is assumed to be in position at station C, whose position has been located by an azimuth traverse and whose elevation is 633.3. The transit is oriented by a backsight on station B, the azimuth being 5°35'. The H.I. is found to be 4.4 ft, and $k = 100.00$.

A reading is then taken on object 1, which is a point on the property line fence. The bottom cross-wire is set on a full footmark and the intercept is read at the upper cross-wire. The middle cross-wire is then set to read 4.4 on the rod, and the instrumentman then motions the rodman forward. While he is finding another point, the transitman reads the vertical angle and the azimuth. He also locates the point on a sketch or describes it in his notes for plotting. For reasons indicated in the article which follows, the azimuths of de-

tail points are read to the nearest 5′ only, and vertical angles to the nearest 1′.

At point 2, it is noted that, to determine its elevation, the rod was read directly as 3.9 ft, and since the H.I. is 4.4, the difference in elevation is +0.5 ft.

At point 5, it was impossible, for some reason, to sight the rod at 4.4, and the point 8.0 was used instead. Accordingly, a correction of 3.6 ft is applied to the calculated difference in elevation.

Before leaving the station a check reading is taken on B and found to be 5°37′. This assures the transitman that the instrument has not been disturbed while his observations were being taken.

The reductions of the notes, shown in the last three columns, are usually made in the office or drafting room.

9-9. Sources of Error *Reading the Rod.*—The inability of the instrumentman to estimate exactly the rod intercept is a source of an accidental error. Its magnitude varies with the weather conditions and the length of sight, but for average conditions the average error will be, perhaps, for distances up to 300 ft, ±0.01 ft, and ±0.01 to ±0.03 ft for distances between 300 and 800 ft. Beyond that distance the error is subject to large variations.

This error is minimized by limiting the length of sight, by repeating the reading, and by the use of targets on the rod.

Rod Not Plumb.—If the rod is not plumb, the intercept is too large and the corresponding distance is too great. This is a systematic error and should be carefully prevented in traverse work. In leveling, this error may affect either a backsight or a foresight and thus becomes compensative in its effect. It may be minimized by using a rod level.

The Vertical Angle.—For inclined sights the vertical angle is used to find both the correct horizontal distance and the difference in elevation. For the former purpose, small errors in measuring the vertical angle are negligible; but for the latter, this source of error is important. Its magnitude can be calculated readily if the sine or tangent of small angles be taken as 0.0003 times the number of minutes. At a distance of 500 ft, the error in elevation due to an error of 1′ in the vertical angle is 0.15 ft, and this value is proportional for other distances.

The Constant k.—Since all rod intercepts are multiplied by the constant $F/i = k$, any error in the determination of this constant is

a systematic error. The resulting error in measured distances is directly proportional to the error in the value of k. Thus, an error of 1% in k will introduce an error of 1% in all measured distances. Accordingly, in comparison with the other errors, it is desirable that this error shall not exceed more than ± 0.1 of 1%.

Length of Rod.—It is obvious that any error in the length of a stadia rod is multiplied by k, or about 100, in the horizontal distance. Thus, an error of 0.1 ft in a 12-ft rod will introduce an error of $0.1 \times 100 = 10$ ft in a distance of 1200 ft. It is therefore important that the lengths of all stadia rods in use should be checked occasionally with a steel tape.

Parallax.—Obviously, parallax, if present, is an important source of error in stadia work. It is to be prevented by keeping the eyepiece accurately focused on the cross-wires at all times.

Natural Errors.—Such natural errors as wind, differential refraction, moisture, and temperature changes affect stadia measurements more or less depending on weather conditions. The two latter effects are usually not important. The wind is frequently troublesome both to the instrumentman and rodman, and accurate results cannot be expected when a high wind is blowing.

Differential refraction is that effect whereby the line of sight as fixed by one cross-wire is refracted more than the line of sight fixed by the other. This condition is caused by the difference in temperature and density of the air strata just above the ground. It is evidenced occasionally by so-called heat waves, but is present in lesser amount at all times. It may be minimized by keeping the line of sight fixed by the lower cross-wire, well above the ground; by reducing the lengths of sights when the heat waves are apparent; or, if practicable, by avoiding stadia work when this condition is serious.

9-10. Errors in Stadia Leveling The effects of the various errors in measuring horizontal distances have been indicated in the previous article, but their effects in determining differences in elevation are not so evident and should be understood.

These effects are illustrated in Fig. 9-6. Let α_1 and α_2 be the measured vertical angles from a given transit station to two different points, A and B; let e_a be the error in each measured vertical angle, which is assumed to be the same for each angle; let e_d be the error in the measured distance ks, which is assumed to be the same for each point. Then E_{a1} and E_{d1} represent the errors in the elevation of

A, due to the errors e_a and e_d, respectively; also E_{a2} and E_{d2} represent the errors in the elevation of B, due to the errors e_a and e_d, respectively. It is apparent that the errors E_{a1} and E_{a2} are of practically the same magnitude, but that E_{d2} is much larger than E_{d1}. In

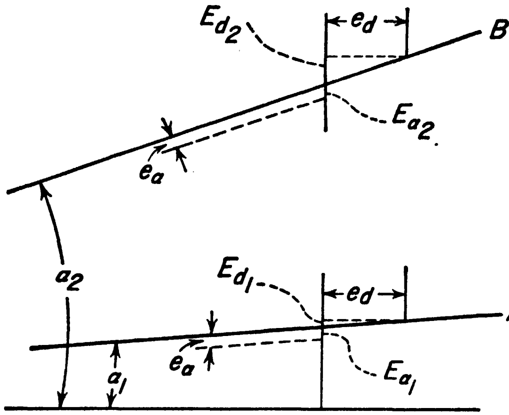


FIG. 9-6. Errors in Stadia Leveling.

other words, a given error in the vertical angle will have about the same effect on the elevation of a point whether the vertical angle be small or large, but a given error in the measured distance will cause a much greater error in the elevation of a point for which the vertical angle is large than for a point for which the vertical angle is small.

EXAMPLES:

(a) It is assumed that an error of 1′ (e_a) is made in measuring the vertical angle of 5° (α_1) to a point A, 500 ft distant. Also, that an error of 2 ft (e_d) is made in measuring the distance.

From Table XIV, $E_{a1} = 0.15$ ft and $E_{d1} = 0.17$ ft.

(b) Assume conditions for point B to be the same as for point A above, but that the vertical angle (α_2) $= 20°$.

Then $E_{a2} = 0.16$ ft, and $E_{d2} = 0.73$ ft.

Hence, it is seen that while E_{a1} and E_{a2} are about the same, E_{d2} is more than three times as great as E_{d1}.

9-11. Checks In transit-stadia traversing, both a backsight and a foresight are taken for each course for the purpose of measuring the angles. Hence, if the stadia distance is read both backward and

forward, these duplicate measurements will provide a check. Another means of checking stadia readings is to read the upper and lower half intervals separately.

9-12. Accuracy *Traversing.*—In transit-stadia traversing, if the horizontal angles are measured to the nearest 1′, the principal part of the total error will be that due to the stadia measurements. The accuracy of these measurements varies greatly with field conditions, as has been shown above; but experience has shown that an accuracy expressed by an error of 1/500 may easily be attained under ordinary conditions; and with careful duplicate readings and limited lengths of sight (perhaps 1000 ft), an accuracy of 1/750 may be expected.

Since a principal source of error is that of reading the rod, and since this is an accidental error, it is evident that the total error is not directly proportional to the length of the traverse, but the effect of compensation will render the ratio of error relatively smaller as the length of the survey increases.

Accordingly, for traverses of several miles in length, an accuracy of 1/1000, and even 1/2000, or better, has been attained.

Leveling.—From a consideration of the sources of error in stadia leveling, the accuracies indicated below may be expected under the conditions specified. It is assumed, of course, that at least ordinary care is taken with respect to all sources of error.

1. ± 0.4 ft $\sqrt{\text{Miles}}$. Fairly level ground so that most level readings are taken on the rod direct, with an occasional vertical angle required. Sights up to 1000 ft.

2. ± 1.0 ft $\sqrt{M}$. Rolling ground with vertical angles up to 5°. Average error in vertical angle not greater than 1′, and average error in distances not greater than 4 ft. Sights up to 1000 ft.

3. ± 2.0 ft $\sqrt{M}$. Hilly ground with vertical angles up to 10°. Average error in vertical angle not greater than 1′, and average error in distances not greater than 4 ft. Sights up to 1000 ft.

Office Problems

9-1. In determining the constant k for a transit, the following rod readings were taken: 1.16, 3.04, 5.27, 7.14, and 8.32. The corresponding distances from the transit station, measured with a tape, were: 118 ft, 311 ft, 534, 725, and 850 ft, respectively. If $(F + c) = 1$ ft, find the average value of k.

9-2. Given the following data: $k = 100.0$; neglect $(F + c)$; two stadia readings are as follows: (a) $r = 6.17$, $\alpha = 2°10'$; (b) $r = 7.35$, $\alpha = 10°13'$. Find H and V by Eqs. (9-7) and (9-8) and compare with values found from Table VII.

9-3. Given the following data: $k = 101.4$, neglect $(F + c)$; H.I. = 4.2; benchmark A has an elevation of 895.6 ft; backsight readings on A, $\alpha = +4°14'$ and $r = 5.82$; foresight readings on B, $\alpha = -6°22'$ and $r = 7.73$. The reading for the vertical angle on B was taken at the 9.0 ft mark on the rod.

Find the elevation of B. (*Ans.* 761.0 ft).

9-4. What is the effect on the calculated elevation of a point if the error in the vertical angle is 1′ and (a) if the distance is 100 ft? (b) if the distance is 500 ft? (c) if the distance is 1000 ft?

9-5. (a) What are the two separate effects on the calculated elevation of a point if the distance is 650 ft, the vertical angle is 4°10′; (1) if the error in the distance is 2 ft; and (2) if the error in the vertical angle is 2′?

(b) Same as (a) except that the vertical angle is 17°20′.

9-6. Given the following data for a line of stadia levels. Find the elevations of the points indicated. Neglect $(F + c)$.

	Backsight		Foresight		Elev.
Station	Rod Int.	Vert. Ang.	Rod Int.	Vert. Ang.	
B.M.$_1$. . . .	6.13	+4°13′			1247.7
T.P.$_1$	5.80	−3°18′	9.16	−3°08′	
T.P.$_2$	3.10	−10°14′	8.24	+4°42′	1253.5 (*Ans.*)
T.P.$_3$	2.14	−18°44′	2.57	+7°04′	
B.M.$_2$. . . .			9.16	+4°07′	1469.8 (*Ans.*)

Field Problem 9-1. Finding the Constant *k*

Procedure.—Set the transit (or plane table) where a distance of about 700 ft can be measured over level ground and, preferably, where the sunlight will fall on the face of the stadia rod. Set four stakes or taping pins, on line at paced distances of approximately 100, 300, 500, and 700 ft. Read the rod at each stake, carefully estimating the interval to the nearest 0.01 ft, being careful that all parallax is removed. Two series of readings should be taken, by different observers, as a check. Measure the distances from the instrument to each stake to the nearest 0.1 ft. Use the mean of the two observers' readings for each distance and calculate the corresponding value of k, to a precision of 0.1%. Thus, four values of k will be found, and these values should agree within 1%, and preferably within 0.5%. The mean of the four values found will be the de-

termined value of k. Record the data in good notebook form, adding a sketch and a paragraph of explanation of the procedure.

Field Problem 9-2. A Transit-Stadia Traverse

Procedure.—Set up the transit at one station of a closed field, and as regards the measurement of the angles, proceed as indicated for either of the methods described in Art. 6-4. It is not necessary to use tacks in the stakes for this work, and the center of the stadia rod may be sighted for direction. The form of notes for a transit-stadia azimuth traverse is shown in Fig. 9-4. The angular error of closure should not exceed 5′ times the square root of the number of sides. The discrepancy between a forward and a backward rod reading should not exceed 1/200.

Field Problem 9-3. Transit-Stadia Levels

Procedure.—Begin with a benchmark and set the transit at any suitable location where good sights can be taken both forward and backward. Level the instrument carefully, set the bottom cross-wire on a full foot mark on the rod and estimate the rod interval to the nearest 0.01 ft. Set the middle cross-wire on a full foot mark and read the vertical angle to the nearest 01′. The rodman then proceeds and selects a suitable turning-point. Again the transitman reads the rod interval, and, to measure the vertical angle, he sets the middle cross-wire on the same foot mark which he used on the backsight. The transitman then moves forward and the procedure is repeated.

If, at any station, it is impossible to use the same foot mark for reading the vertical angle on a foresight that was used on the corresponding backsight, any other foot mark may be used and the proper correction recorded in the notes.

In this work it is important that both the telescope bubble and the vernier of the vertical arc should be carefully adjusted, and that the plate bubbles should be carefully centered. A control bubble on the vertical circle is very desirable. The data of Office Prob. 9-6 will suggest the arrangement for a form of notes.

With a transit in good adjustment, careful readings, and sights limited to, say, 500 ft, the error of closure should not exceed $\pm$ 0.5 ft $\sqrt{M}$.

CHAPTER 10

CONTOURS AND CONTOUR CONSTRUCTION

10-1. Remarks Many studies of natural resources and designs of engineering projects require a representation of the character and relief of the earth's surface. This has been done by different devices including hachures, shading, and models or relief maps; but the device most commonly used for engineering studies is the contour line. To the layman, this method of showing relief is not as plain or legible as others, but it permits the portrayal of the ground surface with greater accuracy, fidelity, and ease than do other methods, and hence the engineer should be thoroughly familiar with the use and construction of contour maps.

In this book the term *contour* will refer to a visible or an imaginary line on the ground, and the term *contour line* will refer to the representation of a contour on a map.

10-2. Characteristics of Contours and Contour Lines

1. A contour is an imaginary line connecting points having the same elevation. It may be thought of as the trace of the intersection of a level surface with the ground surface. The shore line of a body of still water is the best visible example of a contour. A contour line is the line on a map representing a contour on the ground.

2. A *contour interval* is the vertical distance, or difference in elevation, between two adjacent contours.

3. Contour lines spaced closely together represent a steep slope, and when spaced farther apart they represent a more gentle slope. Fig. 10-1.

4. If the ground surface is rough and uneven, the contour lines will be irregular, and, if the ground surface is even, as on earthwork

slopes, the contour lines will be uniform and parallel (see Fig. 10-6).

5. Contour lines which represent summits or depressions are closed lines (see Fig. 10-1). Usually the sequence of adjacent contours, or the presence of a pond or lake, will indicate whether a closed contour

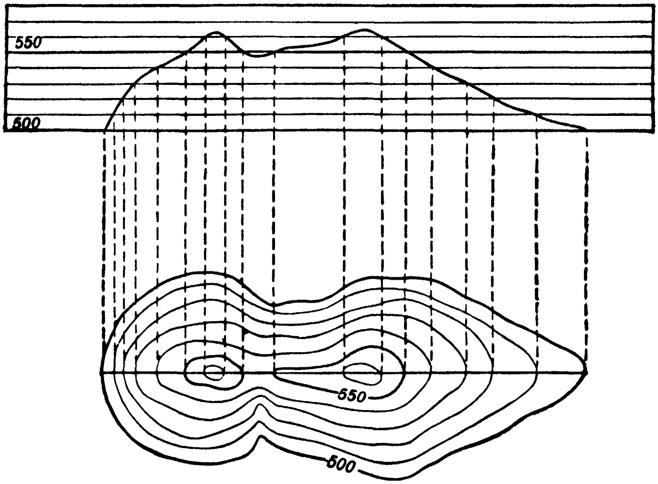


FIG. 10-1. Contour Sketch and Profile.

line represents a summit or a depression. Otherwise a special symbol for a depression contour is used (see Art. 14-6).

It follows that all contour lines are closed lines, either within or outside the borders of a map.

6. Contour lines are arranged in pairs on the two opposite sides of a ridge or a valley line (see Fig. 10-3, *c-2*).

7. Contour lines do not cross each other, nor do any two lines merge and continue as one line.

8. Contour lines are perpendicular to the direction of the steepest slope.

9. A bend in a contour line always points up a valley, or down a ridge.

Figure 10-2 shows both a perspective view and the corresponding topographic map of the same terrain. The principal features include a river, lying between two hills and emptying into a bay of the sea. Nearly all of the characteristics of contour lines listed above are shown in this map, which is drawn to a relatively small scale with a contour interval of 20 ft.

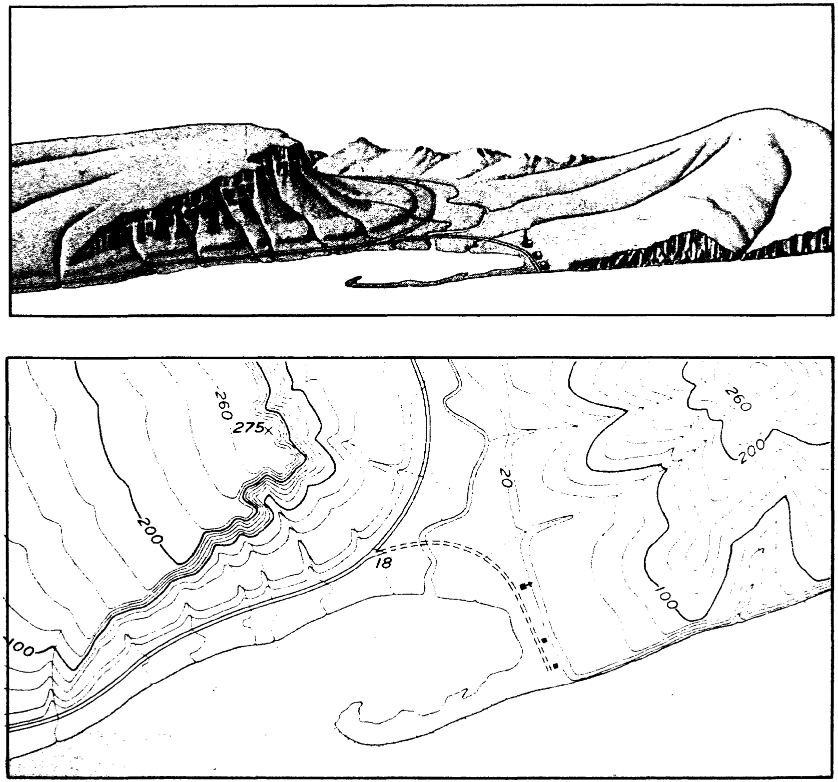


Fig. 10-2. Topographic Map Showing Characteristic Features.

Fig. 10-2 shows both a perspective view and the corresponding topographic map of the same terrain. A principal feature is a river valley lying between two hills and emptying into a bay of the sea. Nearly all of the characteristics of contour lines listed in Art. 10-2 are shown in this map, which is drawn to a relatively small scale with a contour interval of 20 ft. ***Drawn by U.S. Geological Survey***

Drawn by U. S. Geological Survey

10-3. Systems of Contour Points A contour line can be drawn on a map if the plotted position and elevation of properly selected ground points are given. These points may be arranged or grouped in different ways depending on conditions. Four systems are commonly used.

System A. Fig. 10-3, (*a-1*), (*a-2*). System A consists in an established system of squares, usually marked by stakes in the ground. The elevations of the corners are then determined, thus forming a system of coordinate points from which the contour lines may be drawn.

System B. Fig. 10-3, (*b-1*), (*b-2*). If a series of points having the same elevation are located on the ground and plotted on a map, the line joining these points will be a contour line. Thus, if the series of points having the elevation 914 ft are plotted, the 914 contour line is found by drawing a smooth line through these points.

System C. Fig. 10-3, (*c-1*), (*c-2*). Although System B provides an accurate contour map, it requires the location of many points. Where such great accuracy is not necessary, a more expeditious method is that in which a few controlling points only are located and the contour lines are interpolated and sketched in, to represent the ground surface. Such points are summits, depressions, changes in slope, and especially points along ridge and valley lines.

System D. Fig. 10-3, (*d-1*), (*d-2*). To establish System D a transit traverse is first run, with stakes set at 100-ft intervals, over which profile levels are taken. At these points, cross sections are taken to locate the contour points, valley lines, etc. From this system of points the contour lines may be drawn.

10-4. Contour Line Construction For Systems A and C, it is necessary to interpolate between the plotted points to locate the positions of the contour lines. This interpolation may be done by estimation, by calculation, or by graphical means.

1. *Estimation*. The method of estimation is used where the highest accuracy is not desired, where the ground forms are quite regular, and where the scale of the map is intermediate or small. Thus, the contours of Fig. 10-3, (*c-2*), have been interpolated by this method. For example, on the valley line between the two points whose elevations are 1157 and 1193, it is at once noticed that the next higher contour above 1157 is 1160; also, that the additional contours, 1170, 1180, and 1190 fall on the stream line below 1193. Accordingly, the

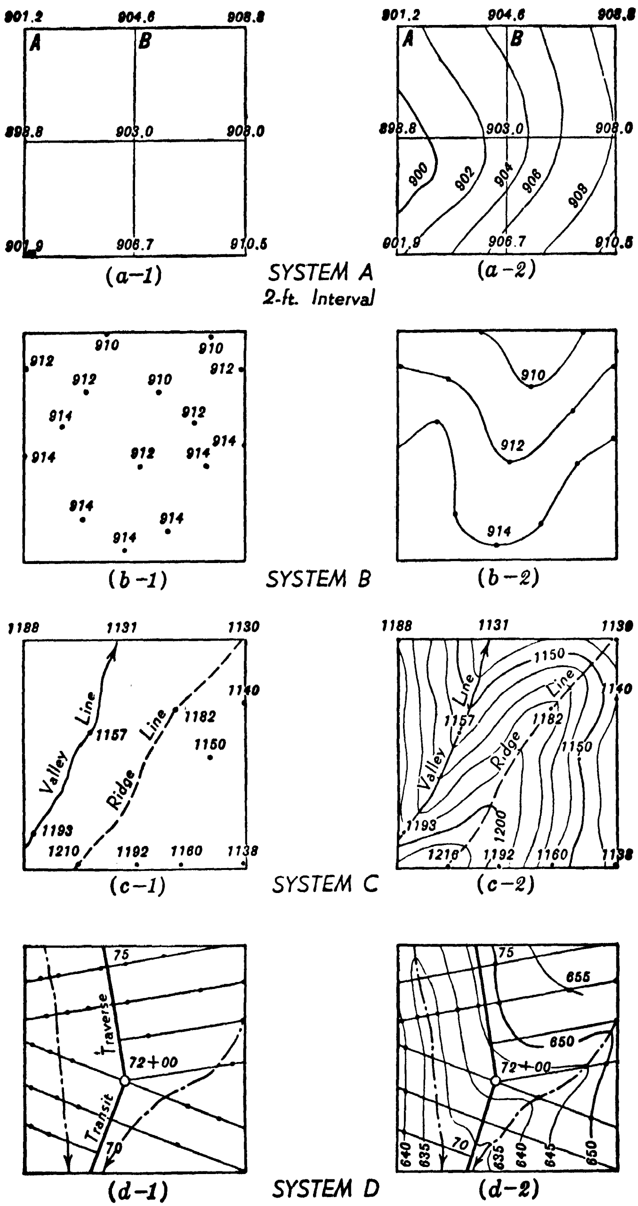


FIG. 10-3. Systems of Contour Points.

four points where these contours cross the stream are spaced by estimation and marked on the map. Possibly the first trial is seen to be erroneous, and a second spacing will be desirable. Similarly, the contour lines are spaced between the other controlling points, after which the contour lines may be sketched in to complete the map.

2. *Calculation.* The method of calculation is used where high accuracy is desired, and where the scale of the map is intermediate or large.

EXAMPLE. It is desired to space the contours between points A and B, along the top line of Fig. 10-3, (*a-1*). It is noticed that the difference in elevation is 3.4 ft; also that the difference in elevation between point A and the next higher contour (902) is 0.8 ft. The distance (d) from A to the 902-ft contour line is given by the relation, $d = 0.8/3.4 \times AB = 0.24\ AB$. If it is assumed that the distance $AB = 100$ ft, then $d = 24$ ft and the 902 contour line is plotted at that distance from A. Likewise, the 904 contour line is plotted at the distance $2.8/3.4 \times 100 = 82$ ft from A. These calculations are readily made on the slide rule.

3. *Graphical Means.* If many interpolations are to be made, and if a relatively high accuracy is desired, it will be more rapid and convenient to provide a proportional scale by which the contour points may be interpolated. Such a scale is drawn on tracing cloth or tracing paper, showing parallel lines (to any convenient scale) to represent the desired contour interval. Figure 10-4 illustrates such a scale drawn for plotting 2-ft interval contour lines.

The scale is used as follows: Suppose A and B are two points,

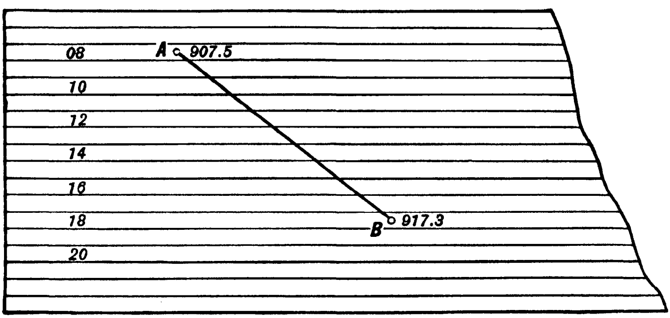


FIG. 10-4. Proportional Scale for Contour Points.

907.5 and 917.3 ft, respectively, and it is desired to interpolate the 2-ft contour lines between these points. The tracing-paper scale is shifted over the points A and B until the point A shows at elevation 07.5 on the scale, and point B shows at elevation 17.3. In this position, the 908, 910, 912 (etc.) contour lines are found at the intersections of the 08, 10, 12-ft (etc.) lines with the plotted line AB. These points are fixed by pricking through the tracing paper into the drawing.

10-5. Generalization Obviously it is feasible to plot only a few points on any given contour line and the line must be drawn freehand between such points. Contours are fairly regular curved lines except for the rough surfaces of outcropping rock strata; contour lines, therefore, are smooth curves which conform to one another more or less closely depending on the regularity of the ground forms. Thus, the lines joining the contour points of Fig. 10-3, (*b-1*), (*b-2*), are not straight but are smooth curves.

In drawing the contour lines on a map, therefore, the draftsman must use his judgment and skill properly to represent the ground surface. If the ground is flat, the scale of the map large, and many points are supplied, the skill required in drawing the contour lines will be slight. But if the ground slopes are steep, the scale of the map intermediate, or small, and if relatively few controlling points are given, then great skill is required.

It is further noted that contour lines at ridge- and valley-line crossings have characteristic shapes and forms depending on the scale of the map and the character of the terrain. For example, the 635-ft contour line of Fig. 10-3, (*d-2*), represents a fairly narrow, flat valley, whereas the contour lines of Fig. 10-3, (*c-2*) represent a much steeper valley, with the depression of the stream much less pronounced. Also, since the ridge and valley lines fix the position and direction of the contour lines, they should be sketched in and the contour lines spaced upon them before the latter are drawn.

It will usually aid in giving the proper character to the contour lines if every *fifth* contour line is carefully sketched in. These lines will then serve as guides when the intermediate lines are being drawn.

10-6. Contour-Map Studies Since a contour map is a representation of the earth's surface in its three dimensions, it provides

the data for an endless variety of uses. For engineering purposes the principal uses of contour maps include representation of (a) reservoir areas and volumes, (b) drainage areas, (c) bridge and building sites, (d) earthwork estimates, and (e) route projects.

(a) *Reservoirs*. In the design of reservoirs for water supply, power, or irrigation projects, the studies are made on contour maps to locate the dam, to determine the volume of water to be impounded, to locate the boundary of the area to be submerged, and to find the drainage area. The necessary maps will be drawn to different scales suitable to the different studies which are made. Thus a large-scale map will be used to fix the dam site, an intermediate-scale map will be used to determine the area and the volume of the reservoir, and a small-scale map will be used to find the drainage area.

The method of finding the area and volume of the water to be stored may be indicated by reference to Fig. 10-5. As the water is impounded and rises by 10-ft stages to the elevations of 1030, 1040, and 1050, it will have as its shore line the corresponding contour lines as shown. It may be noted that each contour line is a closed line within the reservoir area. If the areas of the 1030- and the 1040-ft contour lines are determined with a planimeter, averaged, and multiplied by the vertical distance between them (10 ft), the result is the volume of water included between these contours. Similarly, the volumes between successive contours is computed to determine the total volume of the reservoir.

Obviously, the maximum flood line of the reservoir will be given by that contour having the elevation of the crest of the dam, increased by whatever head of water may exist above it.

(b) *Drainage Areas*. Any drainage area may be traced on a contour map by finding the ridge line around the watershed, as shown in Fig. 10-5. This line may not always be evident at all places on the map, and some field measurements are sometimes necessary. The area is found by planimeter measurements.

(c) *Bridge and Building Sites*. In fixing the location of important structures such as dams, bridges, and buildings, use is frequently made of contour maps. Such maps are drawn to a large scale, from 1 in. = 10 ft to 1 in. = 100 ft and with contour intervals from $\frac{1}{2}$ ft to 2 ft, by means of which the architect or engineer is able to find the best location for his structure.

(d) *Earthwork Estimates*. Earthwork estimates for route projects

are usually made either from profiles or from cross-section notes taken in the field. But earthwork for other purposes is frequently estimated from contour maps. For example, Fig. 10-6 illustrates the

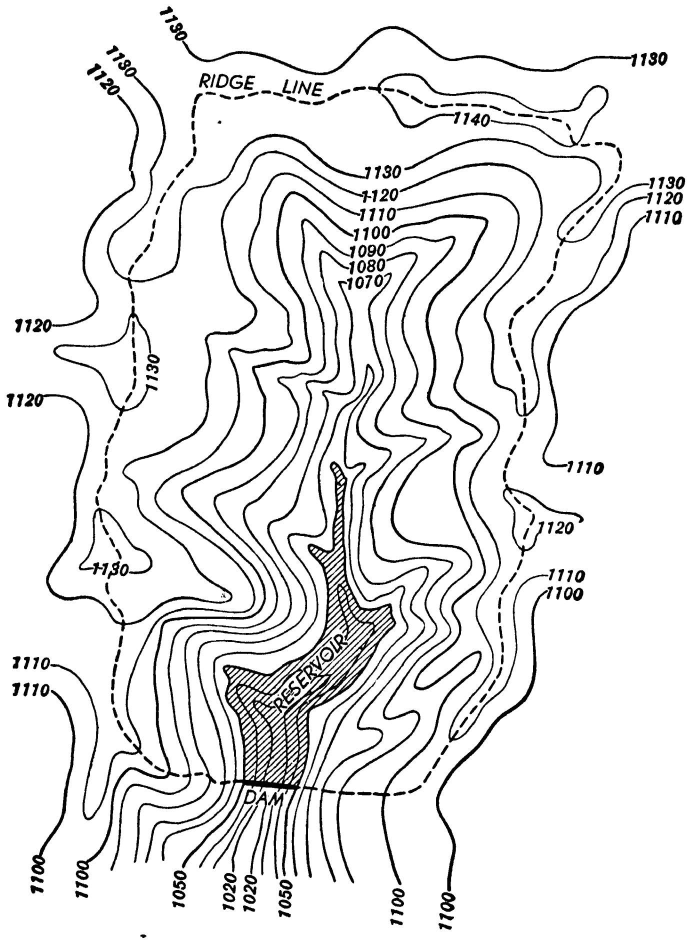


FIG. 10-5. Reservoir and Drainage Area.

method of estimating the earthwork necessary to construct a parking space on a hillside.

The irregular lines are the contour lines of the original ground surface; the straight and circular lines are the contour lines of the proposed earthwork. The conditions are as follows: the area of the

parking space is indicated by the heavy line rectangle; the elevation of the surface is 908 ft; the side slopes are 3 to 1.

From these conditions and from the principles of contours, the contour lines representing the proposed earthwork are drawn as shown. Since the elevation of the parking space is to be 908 ft, the contour lines above that elevation represent cut (shown cross-hatched) and those below that elevation represent fill.

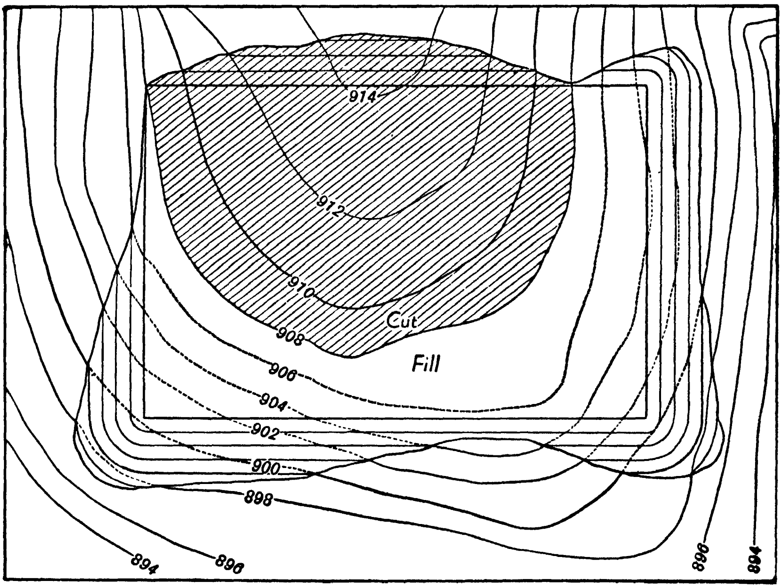


FIG. 10-6. Earthwork Estimate.

The volume of cut is found by determining with a planimeter the area of the closed 908-ft contour line in cut and that of the 910-ft contour line. The volume of earthwork between these two contours is then found as the average of the two areas multiplied by the contour interval, 2 ft. Similarly, the volume between the 910- and 912-ft contours is found, etc.

The volume of fill is found in a manner similar to that for the cut. It may be noted that, because of the ridge line through the parking space, the area of the 904-ft contour line is divided into two parts—one on the left and one on the right of the ridge. Also for the contour lines below 904 ft. The results may be recorded as shown below.

CUT

Contour Line	Area sq in.	Area sq ft	Volume cu ft
908	10.15	25,400	
			42,300
910	6.78	16,900	
			25,200
912	3.32	8,300	
			9,600
914	0.52	1,300	
			77,100 = 2,850 cubic yards

FILL

Contour Line	Area sq in.	Area sq ft	Volume cu ft
908	7.70	19,200	
			32,700
906	5.40	13,500	
			23,100
904	3.94	9,600	
			16,300
902	2.68	6,700	
			9,800
900	1.24	3,100	
			3,600
898	0.22	500	
			85,500 = 3,170 cubic yards

(e) *Route Projects.* Studies for the location of such route projects as railways, highways, and canals are frequently made on contour maps. Many conditions govern such locations which need not be discussed here, but that one for which a contour map offers special aid is the establishing of a uniform or a maximum grade. The procedure may be described by reference to Fig. 10-7, where two exist-

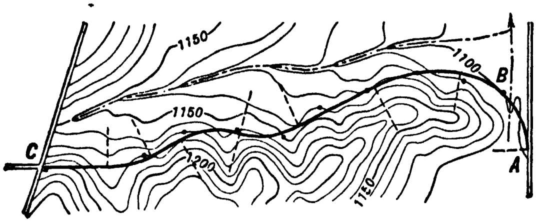


FIG. 10-7. Route Location.

ing highways are shown and it is desired to locate the centerline of a proposed connection between points A and C. The difference in elevation and the distance between these points makes it desirable to establish a uniform grade of approximately 2.5%.

Since the contour interval is 10 ft, it is evident that a uniform grade of 2.5% will rise from one contour to the next in a horizontal distance of 400 ft. Accordingly, if the feet of a pair of dividers are set at a map distance of 400 ft apart and if one foot is placed on one contour and the other foot on the next adjacent contour, the line joining these two points will represent a grade of 2.5% on the ground. If the feet of the dividers are then turned about one foot, and the other placed on the next higher contour line, a third point on the uniform 2.5% grade will be found, and so on. By this means a series of points is located which, if joined by a smooth curved line, would represent a uniform grade of 2.5%. Such a line is sometimes called a *grade contour*. The location of the highway that will require the least earthwork will be that one which conforms most closely to the series of points thus located. Obviously the limitations as to curvature prevent the location from passing through all of the points, but it is made to conform to them as nearly as may be.

Thus, in the figure, beginning at point B, a series of points (round dots) have been located as described above, to fix the 2.5% grade contour; and the location, shown by the full line, has been made to follow the grade contour as closely as the conditions of alignment and ground forms would permit.

Obviously, this solution is not definite, and after a first location has been made another trial may show a better one. The process is not carried beyond two or three trials because the final adjustment is necessarily made in the field.

Office Problems

10-1. The situation at a railway bridge site is as follows: elevation of surface of the stream, 585 ft; elevation of the base of rail, 601 ft; the grade of the railway in each direction is 1%; and the slope of the ground each way from the stream is about 6%. The bridge is 100 ft long.

Draw a contour map of this location extending 600 ft each way from the bridge site, using a contour interval of 5 ft.

10-2. The conditions at the intersection of two paved streets are as follows: elevation of center of intersection, 517.00; gradient of the east-west street, −3.00% to the east, and of the north-south street, −1.00% to the north, the pavements are 28 ft wide; the streets

are 60 ft wide; the sidewalk grades are about 0.5 ft above the curbs; the curbs are 7 in. above the gutter, and the crown is 4 in.

Draw a contour map of the intersection showing the 516- and the 518-ft contour lines.

10-3. A water supply reservoir has been constructed for which the conditions are as follows: A stream bed slopes toward the south on a 4% gradient; the dam is of earth, its crest being 20 ft wide, elevation, 932; elevation of the stream bed at the dam is 907; the upstream slope of the dam is 5 to 1, and the downstream slope is 3 to 1; the east bank of the stream has a 10% slope and the west bank has a 17% slope.

Draw a contour sketch of this reservoir, using a 5-ft contour interval and a scale of 1 in. = 100 ft.

10-4. Estimate the volume of water impounded in the reservoir of Prob. 10-3 when it stands at the 930-ft stage.

10-5. Estimate the volume of earthwork in the dam of Prob. 10-3.

10-6. A gravel pit on a hillside is roughly circular in shape and about 200 ft in diameter. The slope of the hill was originally 8 per cent. The elevation of the lower edge of the pit is 941 ft, the elevation of the bottom of the pit is 935 ft (about 50 ft inside of the lower edge), and the elevation of the top edge is 957 ft. Draw a sketch of the pit using the even-numbered 2-ft contour lines and a scale of 1 in. = 100 ft.

CHAPTER 11

THE PLANE TABLE

11-1. Remarks The plane table is a field-mapping instrument, and for surveys of which the purpose is to construct a map only, it has important advantages. However, nothing is accomplished with the use of a plane table that cannot be done with a transit and drawing-room methods. Many factors offset the advantageous use of one method or the other and these must receive due consideration in choosing the best method for a given survey.

The principal characteristic of the plane table as compared with the transit is that in the use of the latter instrument, the directions of lines are measured with the graduated circle and vernier, recorded in a notebook, and later plotted by drawing-room methods; whereas, in the use of the plane table, the directions of lines are drawn directly on the plane-table sheet in the field, at the time each object is sighted.

The complete plane-table instrument includes the alidade, the drawing board, and the tripod, as illustrated in Fig. 11-1.

11-2. The Telescopic Alidade The telescopic alidade is illustrated in Fig. 11-2. It consists of a telescope mounted on a column which is fixed to a steel straightedge about $2\frac{1}{2} \times 15$ in. The telescope is similar in all respects to the transit telescope and is supported by a horizontal axis which rests on short standards above the supporting column. By this arrangement the telescope may be rotated in a plane perpendicular to the horizontal axis through a given arc. A vertical arc, vernier, clamp, and tangent screw are provided; also a striding, or an attached level, which permits the instrument to be used for direct leveling. A compass needle is usually provided, housed in a small metal box on the straightedge blade. A small bubble tube

having a spherical surface, placed on the blade, is used to level the table. In some instruments two bubble tubes, one at right angles with the other, are used for this purpose. A third bubble tube mounted on the vernier frame facilitates the measurement of vertical angles.

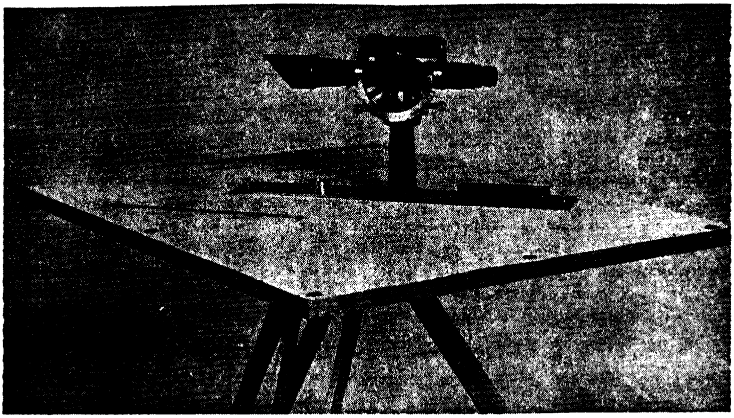


FIG. 11-1. Plane Table Equipment.

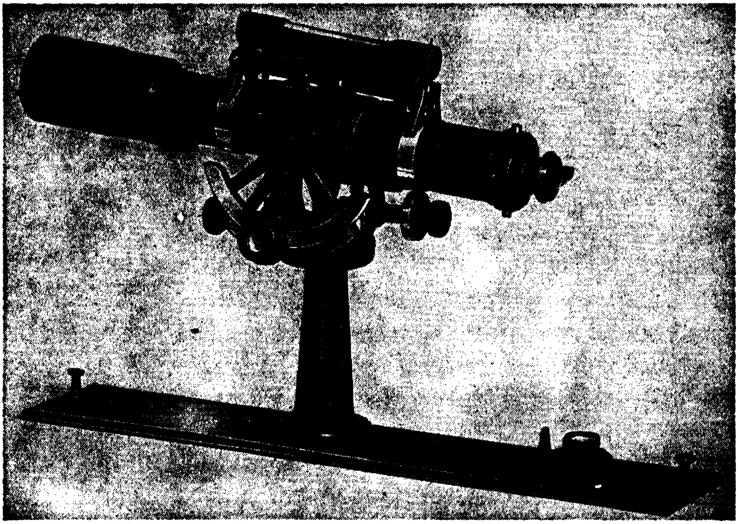


FIG. 11-2. Telescopic Alidade.

11-3. The Johnson Type Table The Johnson-type table movement is illustrated in Fig. 11-3. This mechanism supports the table and alidade shown in Fig. 11-1. The drawing board is a plane board being made in different sizes for different field conditions. The most

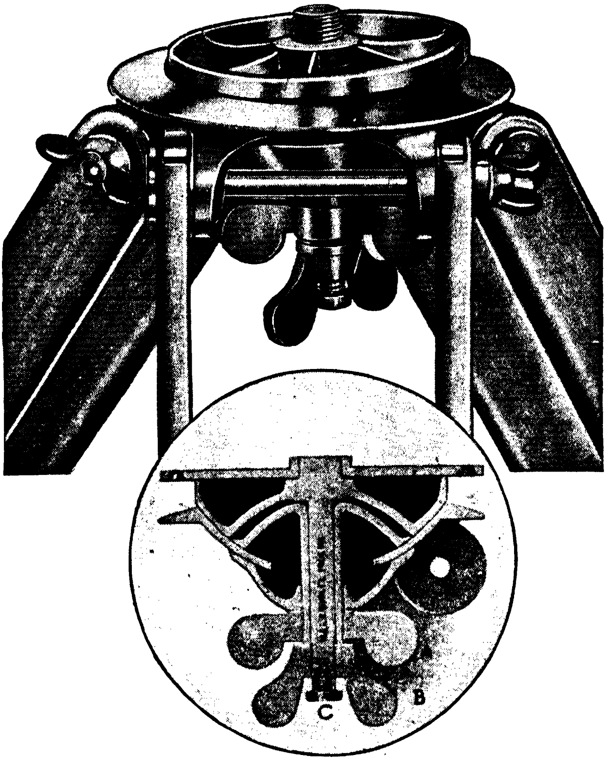


FIG. 11-3. Johnson Table Movements.

common size is 24 × 30 in. A flat metal disc is let into the underside of the board by which it is screwed onto a threaded bolt of the tripod, thus securing the board to the tripod.

Sometimes the compass needle is placed in a metal box set into the edge of the drawing board, instead of being fixed to the alidade straightedge.

The tripod is provided with a special device by which the board is first leveled and then rotated, thus orienting the instrument for field use.

11-4. Accessories Since the map is drawn in the field, certain accessories are usually a part of the field equipment, including drawing paper, pencils, scale, and tables (see Table VII) or slide rule for reducing the stadia readings.

The drawing paper is subject to rough usage and exposure to the weather. It should, therefore, be durable and resistant to weather changes. A sheet of high quality paper mounted on muslin makes a suitable sheet. If the construction of the map extends over a period of several days, a cover sheet of smooth tough paper is used to protect the plane-table sheet. A portion of the cover sheet is torn away to expose the drawing as the work progresses. A pencil of not less than 6H hardness, finely sharpened to a chisel edge at one end and a round point at the other, is used.

A stadia slide rule is much more rapid in use than are tables. For most surveys the 10-in. rule is sufficiently accurate. The calculations necessary in the use of stadia tables will be facilitated by the use of the ordinary slide rule.

11-5. Setting Up and Orienting the Plane Table In plane-table surveys, the locations of lines and points are plotted directly upon the drawing paper, and accordingly, a pencil dot on the paper will represent a small or a large dimension on the ground according to the scale to which the map is drawn. For example, a fine pencil-point dot may be 0.01 in. in diameter. If the scale of the drawing is 1 in. = 100 ft, evidently such a dot represents an area 1 ft in diameter on the ground. Thus it is evident that in centering the drawing board over a ground station, it is not necessary to use the care that is required in setting up a transit. Usually the table is centered over the ground point by estimation only. However, on careful work and if the scale of the map is large, e.g., 1 in. = 20 ft, a plumb bob may be suspended from the underside of the table, thus to center a plotted point over the corresponding ground point.

In setting up the table, it is first turned into its proper relation to the field being surveyed, and placed over the ground station as indicated above. The tripod legs are firmly planted in the ground and spread at such an angle as to bring the drawing board to a comfortable height. This height should be such that the observer can work easily on all parts of the drawing without leaning against the board. The table is then leveled by whatever devices are provided for that purpose.

As the transit must be oriented by means of a backsight, so the plane table must be properly oriented before any mapping is done. In small scale mapping and rough surveys, this may be done by use of the compass needle, but on most work a backsight is used as is the case with the transit.

If the compass needle is used, a line representing the magnetic meridian is drawn on the plane-table sheet, at the initial station. The alidade straightedge is then placed along this reference meridian and the table is turned in azimuth until the needle indicates north. The table is then clamped in this position and mapping proceeds. At the next instrument station, after the table is set up and leveled, the board is again oriented as explained above.

If a backsight is used to orient the table, the procedure is as follows: Suppose the instrument is at station B and is to be oriented by a backsight to station A, Fig. 11-4. After the table is set up and

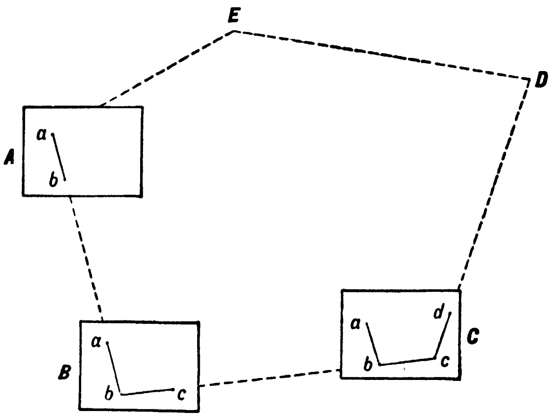


FIG. 11-4. Orienting Plane Table by Backsight.

leveled, the alidade is placed along line ba and turned in azimuth until the observer, looking through the telescope sights station A. The board is then clamped and thus properly oriented.

11-6. Traversing With the Plane Table It has been shown that in transit work the location of instrument stations is effected by the use of traversing or triangulation; likewise, in plane-table surveys the instrument stations may be located by traversing or by *graphical triangulation*. Also, the plane table makes convenient the

use of two other principles of locating points; namely, *intersection* and *resection.*

The procedure of traversing with the plane table is simple and is illustrated by Fig. 11-4. Having chosen a suitable scale for the map, a point (a) is chosen arbitrarily to represent the initial ground point, A, of the traverse. The table is then set up and oriented so that the sheet has a proper relation to the field. The straightedge of the alidade is then pivoted about the point, a, on the map and a sight is taken toward the forward station B, and a line is drawn along the straightedge in that direction. The distance AB is measured either by stadia or tape and the corresponding map distance is sealed to locate point b.

The table is then moved to station B, set up, and oriented by a backsight on A as explained in the previous article. A foresight is then taken on C by pivoting the alidade about b, sighting C, and drawing a line on the map in that direction. The distance BC is then measured, and point c plotted on the map. In this manner the traverse proceeds, and if the field is a closed one, the final position of a should fall on the initial position. The error of closure is, therefore, at once apparent as soon as the last observation is made.

If the error of closure is within proper limits, it is adjusted by methods explained in Art. 14-13; otherwise, the work is repeated until the error or mistake is found.

11-7 The Principle of Intersection The principle of intersection is a simple application of the condition that the position of a point is fixed by lines of direction drawn toward it from any two other known points. The method is especially convenient in plane-table work. Thus, in Fig. 11-5 let A, B, C, represent instrument stations, located by the traverse ABC. At station A direction lines (*rays*) are drawn from a to two distant objects E (a fence corner) and D (a tree). At Station B rays are again drawn from b toward E and D. These rays intersect those previously drawn from station A, and these intersections fix the positions e and d, being the plotted positions of E and D. When a third station C is occupied and rays again drawn from c toward E and D, if these rays pass through the points previously located then the plotted positions e and d are proved or checked.

It will be noticed that these points are thus located without the necessity of any distance measurements or of a rod being held there.

Obviously, any number of details like E or D, which are visible from two or more stations on the traverse ABC, can be located in this manner.

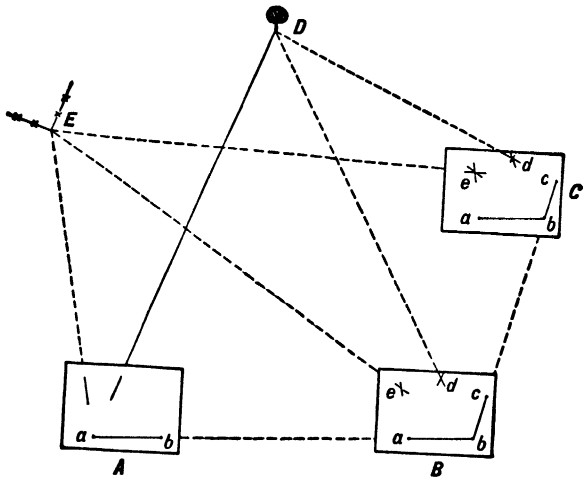


FIG. 11-5. Principle of Intersection.

11-8. The Principle of Resection The principle of intersection as described in the preceding article is useful in locating a distant point from known plane-table stations. The principle of resection is opposite to that of intersection; i.e., it is used to locate the plane-table station from known distant points.

In mapping details it is frequently desirable that the instrument should occupy an advantageous point which has not previously been located and, therefore, is not plotted on the map. The plane table offers a number of solutions for this problem, but it will be sufficient here to describe a simple one only.

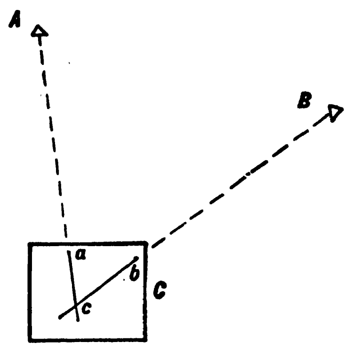


Fig. 11-6. Principle of Resection.

Referring to Fig. 11-6 it may be supposed that it is desired to locate the position of C from sights on two points which have previously been located or plotted on

the map. Also, it is supposed that A and B are two adjacent stations or objects whose positions have been located and plotted at a and b, respectively.

The procedure of locating point c on the map and orienting the board is as follows: The plane table is set up at station B and a ray is drawn toward station C, but no distance is measured. Next, the instrument is set up at C and oriented by a backsight along ray cb, toward station B. Then the alidade is pivoted about point a on the map until object A is sighted in the field, then ray ac is drawn to resect ray bc at point c. This fixes the position of c on the map as being the located position of station C over which the instrument is set up. This procedure is called location by *resection.*

11-9. Graphical Triangulation The method of establishing instrument stations by triangulation with a transit has been explained in Art. 6-8. In a similar manner horizontal control may be extended with the plane table by a method that is called *graphical triangulation.*

The stations are carefully selected, signals erected, and if necessary platforms are built as in the case of transit triangulation. However, instead of a base line, graphical triangulation begins with three known points, accurately plotted on the plane-table sheet. The observer then occupies each of these stations and intersection rays are drawn toward other signals whose positions are to be located. From these three stations a number of additional stations will be fixed by three intersecting lines as described in Art. 11-7. From these additional stations, still others will be located and so the system is extended to cover any desired area.

As the work proceeds many objects will be located other than prepared signals, such as lone trees, church spires, house chimneys, and other definite landmarks. These objects may never be occupied by the plane table, many cannot be, but they will aid greatly in the subsequent work of locating details and many of them may serve for three-point locations of the plane table, when details are being mapped.

11-10. Locating Details From what has been said about the plane-table instrument and methods thus far, it will be a simple matter to describe its use in locating details. The instrument is set up at a given station and oriented as has been explained. The stadia

method is universally used in this work, although on maps to a large scale, certain definite and important objects such as lot corners, water hydrants, sewer manholes, etc., may be located by taped measurements.

The rodman presents his rod at a given object to be mapped. The observer pivots the alidade about the plotted position of the station occupied, sights the rod, and draws a short portion of a ray toward it. He then shifts the alidade to a convenient position on the table, reads the stadia intercept, and then observes either a vertical angle or a direct, level-rod reading to determine the elevation. The computer makes whatever reductions are necessary, and the observer scales the distance to plot the point, writing the computed elevation beside it. Meanwhile, the rodman chooses another point and the work thus preceeds.

As the detail points are plotted, the observer draws the map, sketches the contours, and by suitable symbols represents the features of the area to be mapped. Thus the map is drawn, in all its essential details, in the field while the terrain is in the view of the observer.

11-11. Adjustments of the Plane-Table Alidade The following treatment of the adjustments of the plane-table alidade presupposes a familiarity with the adjustments of the level and transit (see Arts. 3-22 and 5-12).

As regards the mounting of the telescope, alidades are of two kinds. In one, the telescope tube is fixed to the horizontal axis the same as that of a transit. In the other, the telescope tube, near the middle, fits accurately within a sleeve which, in turn, is attached to the horizontal axis. The former may be termed the *fixed-tube;* and the latter, the *tube-in-sleeve* type. In the latter type, the telescope may be rotated in its sleeve just as the telescope of the wye level may be rotated in its wyes. However, as regards the adjustments, no distinction need be made. In the use of the instruments, certain instrumental errors can be eliminated with the tube-in-sleeve type as will be explained.

The striding level is the bubble tube supported by inverted wyes which rests on accurately turned shoulder rings provided on the telescope tube. The striding level can readily be removed or turned end for end on its supports.

1. *Adjustment of the Vertical Cross-wire.* The relation, test, and

adjustment of the vertical cross-wire are the same as for the adjustment of the cross-wire ring in a transit. Art. 5-12.

2. *Adjustment of the Blade Bubble Tubes.* If the straightedge blade is equipped with two bubble tubes, they are tested and adjusted in the same manner as the plate bubble tubes of a transit, Art. 5-12, except that in reversing the bubble tubes, the table is not turned, but the alidade is lifted and turned end for end.

3. *Adjustment of the Striding Level.*

Relation.—The axis of the bubble tube should be parallel to its line of supports.

Test.—With the striding level in place, level the bubble carefully. Remove the striding level, turn it end for end, and replace it. If the bubble tube is in adjustment, the bubble will remain centered after reversal. If not, the apparent error is double the actual error.

Adjustment.—After reversal, bring the bubble back halfway by means of the adjusting screw on the bubble tube. Re-level and repeat the test.

4. *Adjustment of the Line of Sight.* The relation and test are the same as for the adjustment of the line of sight of the level (see Art. 3-23).

Adjustment.—Center the bubble of the striding level and adjust the horizontal cross-wire to the correct reading on the rod.

5. *Adjustment of the Vertical Arc.* The relation, test, and adjustment are the same as for the vertical arc of the transit (see Art. 5-12).

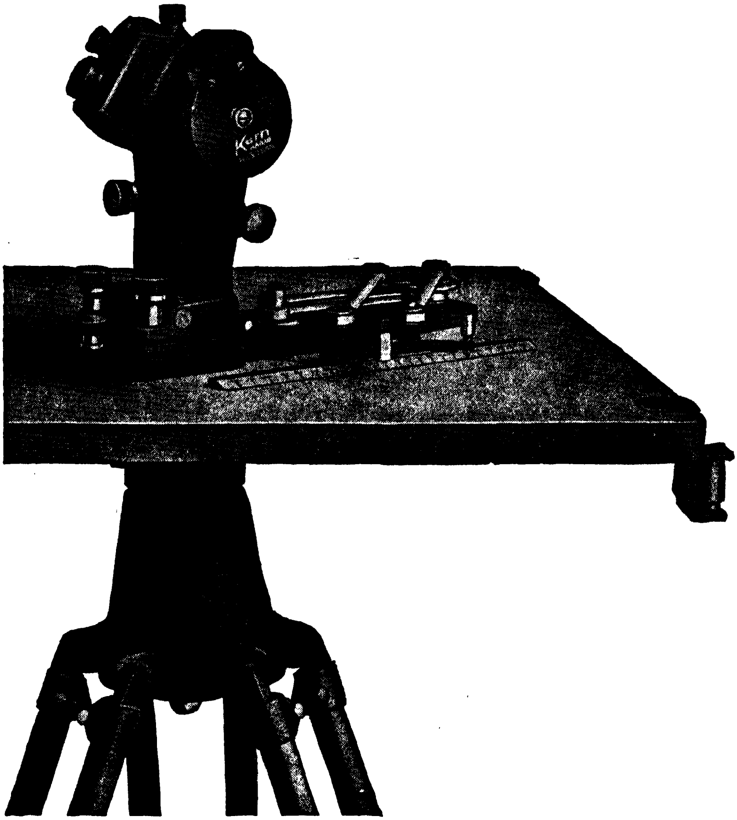
6. *Adjustment of the Vernier Bubble Tube.*

Relation and Test.—When the striding level bubble is centered and the vertical arc reads zero, the vernier bubble should be at the center of its tube.

Adjustment.—Bring the bubble to center by means of the adjusting screw on the bubble tube.

11-12. New Plane-Table Alidades In recent years significant improvements have been effected in alidade design. The most modern types have self-indexing vertical arcs which operate automatically on the pendulum principle. The vertical arc graduations are on glass and are read through an eyepiece situated alongside the telescope. The blade is frequently fitted with a parallel ruler in order to facilitate plotting. Furthermore, some alidades like the one in Fig. 11-7 are self-reducing stadia instruments. The intercept on

the stadia rod between curved lines seen in the field of view of the telescope is multiplied by a factor. This product is either the horizontal distance or the difference of elevation, depending upon what curved lines were used.



Courtesy of Kern Instruments, Inc.

FIG. 11-7. Self-Reducing Alidade.

11-13. Sources of Error In general, the sources of error which affect plane-table work are those which affect transit and drawing-room work and need not be discussed here; but three or four sources merit some attention.

Setting Over a Point and Sighting.—As stated above, the scale of the drawing is usually such that great care taken in centering over a point would be a waste of time. Likewise, in sighting objects, a

small error of 0.2 or 0.3 ft in observing the range pole or stadia board, even on traverse work, is negligible when the errors of plotting are considered. Many objects sighted need not be observed with great accuracy, such as trees, banks of streams, unimproved roads, etc.

Drawing.—The errors of plotting points and drawing a map are always present in addition to the observational errors. Hence, the accuracy of a finished map is directly affected by the care and skill used in its preparation. Many observers use needle points for plotting stations; otherwise, hard (6H or 8H) pencils, carefully sharpened, should be used to insure permanency and accuracy in the map.

Instability.—Because of its size and construction, the plane table is not as stable as the transit. For this reason the observer should be careful not to lean against, or bear unduly upon, the table while observing or drawing. The vernier control bubble is a practical necessity for the proper measurement of vertical angles. With careful and skillful handling, the plane-table will be found to be satisfactory for any desired precision.

11-14. Checks One of the advantages of the plane-table method is the convenience with which checks can be applied. Whenever the table is set up and correctly oriented, the direction on the map to any plotted point should coincide with the direction to the corresponding object in the field. For example, in Fig. 11-4, when the table is in position at station C, if the alidade is directed along line ca, the line of sight should strike the signal at A. The same is true for any plotted point for which the corresponding object is visible. Also, when the traverse is closed in the field, it should also close on the map. Thus, many convenient checks are constantly available as the work proceeds.

11-15. Plane-Table and Transit Methods Compared As compared with the transit, the plane-table method has these advantages: (1) the map is drawn while the terrain is under the view of the observer, and consequently a more faithful and complete representation results; (2) no horizontal angles are measured, no notes are recorded, and no descriptions of objects sighted are necessary, thus saving much time and avoiding many chances for mistakes; (3) omissions in the field work are hardly possible since they must be detected as the map is drawn; (4) many convenient checks are constantly available; and

(5) fewer points need be observed and the total time for completing the map is greatly reduced.

On the other hand, the plane-table method has these disadvantages: (1) the instrument is heavy and cumbersome in the field and in transport; (2) it cannot be used in adverse weather both as to cold and moisture; (3) more skill is required on the part of the observer; and (4) more time is required in the field than is the case with the transit.

As compared with the plane-table method, the transit has these advantages: (1) measurements can be made more rapidly; and (2) greater accuracy in locating definite points is possible.

Comparing these advantages and disadvantages on both sides, the relative merits of the two methods may be summarized in the general statement that, if a survey has for its purpose the location of many definite points, as for example, the buildings, trees, poles, hydrants, etc., of a city subdivision, the transit method may be superior; but, if a map is to be drawn showing many contours, streams, and other general features, the plane-table method is decidedly superior both in economy and in the quality of results.

Field Problem 11-1. Campus Map With a Plane Table

Procedure.—Establish suitable instrument stations over the area such that all points can be observed and that no sights shall be more than 600 or 800 ft long. Connect these stations with a plane-table traverse according to the procedure indicated in Art. 11-6. Adjust the error of closure as explained in Art. 14-13, draw small circles about these points and, while mapping, do not draw any lines through them.

If the terrain is flat, the elevations of the instrument stations should be determined by a line of differential levels having a permissible error of closure of perhaps $\pm 0.2\sqrt{M}$. If the terrain is hilly, the elevations of the instrument stations may be determined by stadia leveling (see Art. 9-7).

The table is then set over a station, oriented, and details are located by the methods of Art. 11-10. Only a short length of each direction ray should be drawn, so as not to render the map illegible by many long lines drawn upon it. As soon as a contour point is plotted, the elevation should be recorded beside it.

Three intersecting rays should be used frequently to locate given points, thus to check the work as it proceeds. Also, as a further check, two or three points should be located at each station such that they can be observed from the next adjacent station. One additional instrument station should be located by resection, Art. 11-8, and one by graphical triangulation, Art. 11-9.

If the vertical arc is equipped with a control bubble, it should be centered just before a vertical angle is read. Otherwise, care must be taken to note the index error for each vertical angle observed.

Complete all essential plotting and sketching in the field, after which the map is finished in the office by the use of proper symbols, a meridian, and title as described in Chapter 14.

CHAPTER 12

LAND SURVEYING

I. Development of the United States Public Lands System

12-1. General Remarks From earliest times it has been the purpose of land surveys to establish and maintain boundary lines. These lines limit property rights; therefore, all land surveys are subject to legal principles that are of ancient origin. Hence, the present land surveyor, while he uses modern instruments and methods, must understand these legal principles which govern this work.

No single description of the United States public lands system will apply to the many states where it has been used, because the system has undergone many changes since it was instituted in eastern Ohio. From time to time the U.S. Land Office and its successor, the Bureau of Land Management, have issued instructions under which the work of subdivision has been done, but many changes have been made as the work progressed. These instructions in the early years were in the form of circulars and letters to the deputy surveyors, and it is now impossible to recover all of them, although careful search has supplied many. In more recent years the U.S. Land Office and the Bureau of Land Management have published "Manuals of Instruction" which have prescribed the procedure to be used in the western states where subdivision work has been, and still is, in progress. But it should be noted that the manner of establishing the land lines in practically all of the middle-western and southern states was very different from that described in any recent manual.

By an Act of Congress March 1, 1800, it was prescribed that the boundaries of the public lands surveys as they were originally established are unchangeable. Accordingly, it is very important that the land surveyor should know the exact procedure by which the

boundaries were established in the locality where his present surveys are made; and in most of the states where the system now exists, he will be seriously misled if he should suppose that the procedure described in the latest Government manual is to be followed in relocating boundaries.

The subject of land surveying is discussed in this chapter under two general divisions: first, the physical aspects of land surveys, especially the development of the rectangular system as it has been applied to the public lands; and second, the legal principles which govern the work of the land surveyor. Some of this discussion, of course, will not apply to the states where the rectangular system does not exist, but much of the discussion, both of the physical and of the legal aspects, is applicable to all surveys whether under the Public Lands System or not.

12-2. Historical The boundary lines of private property established in the colonies consisted principally of such natural features as tidewater shore lines, streams, highways, fences, trees, and stones. The tracts thus bounded were irregular in shape, except for the subdivisions within urban limits. There was no general system to serve as a control for the positions of these boundaries, and, accordingly, when they became obliterated by the passage of time or the construction of improvements, it was often difficult to restore them. Moreover, the deed descriptions of such boundaries were complex and subject to many errors of interpretation and identification. These undesirable conditions prevailed over most of the eastern territory of the United States when the rectangular system of subdividing the public lands was instituted; and because of the impracticability of a change where so many boundaries would be affected, the U.S. rectangular system was never applied to many of the eastern states.

As soon as all claims by the several states to western territory had been ceded to the Continental Congress, a committee was appointed by that body to arrange for subdividing and disposing of the public lands. Thomas Jefferson was made chairman of the committee; hence, he has generally been regarded as the originator of the public lands system. His committee submitted to the Congress in May 1784 a plan for subdividing and disposing of the public lands. This plan was considered, revised, and adopted as an ordinance on May 20, 1785.

The Act of March 1, 1800, provided that corners which have been established on government surveys shall stand as the true corners whether they have been correctly located or not.

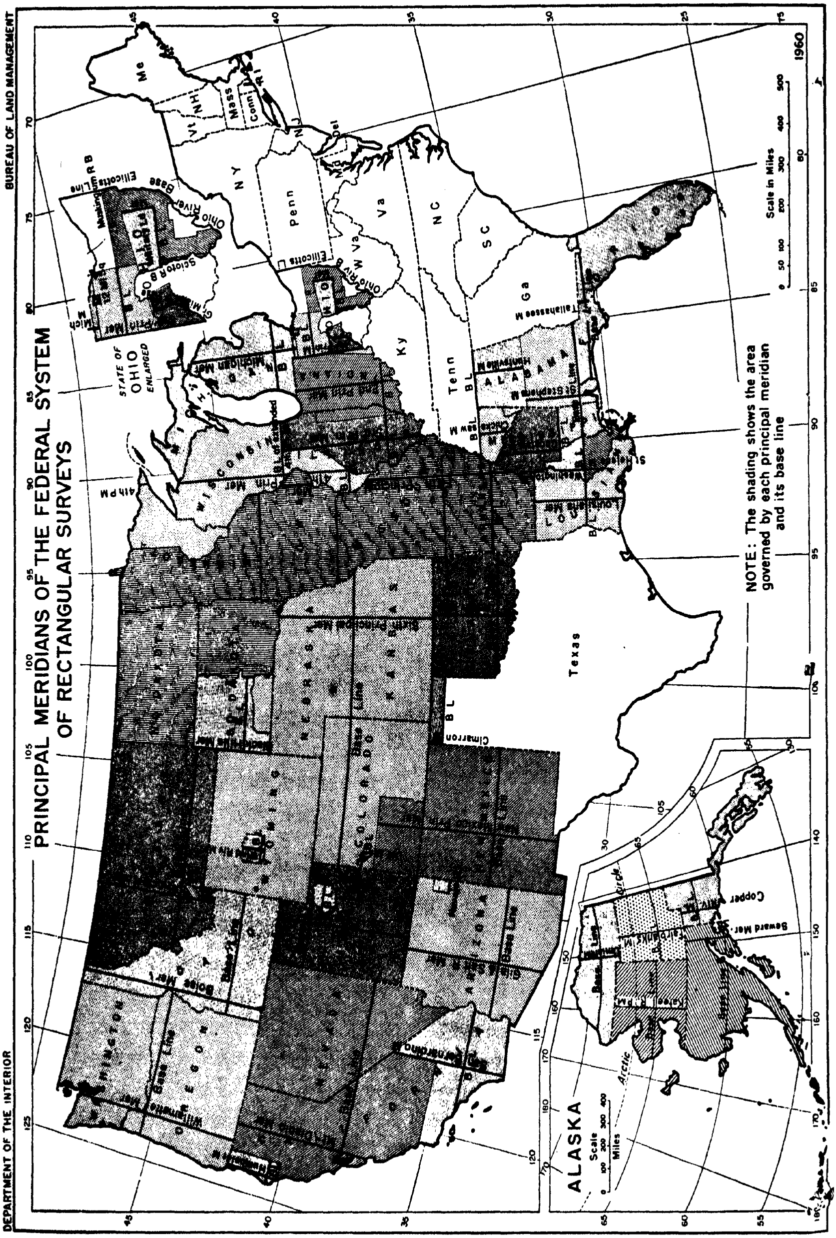
The Act of May 10, 1800, provided (1) for the subdivision of sections into half sections by setting corners at every half-mile on all east-west section lines; and (2) that all excess or deficiency in measuring through a township from east to west was to be placed in the westernmost half-mile, and in measuring from south to north, in the northernmost half-mile.

12-3. Extent of the Public Lands Surveys The territory of the United States that has been or is being surveyed under the rectangular system includes thirty states, as shown in Fig. 12-1. Also shown are the several principal meridians with their corresponding baselines. The shading shows the areas governed by each principal meridian and its baseline.

The rectangular system has been modified, to some extent, in each of these states by grants from Congress to individuals, grants to Indian tribes, or claims to mineral lands; and in the southern and western states, by early grants from the governments of France and Spain. Many of the Indian treaties included grants of land to individuals, both American and Indian; also, many individual property rights had become established, either by purchase or occupation, in the states west of the Mississippi prior to the date of the Louisiana Purchase. The United States system has never been applied in the state of Texas, although a modified rectangular system has been used by that state in subdividing tracts as they existed when it was admitted to the Union.

12-4. Instruments and Methods. Directions The measurements of directions and distances in the early surveys were very crude. The magnetic compass, Fig. 12-2, was the universal instrument used for establishing the directions of all lines, including the principal meridians and baselines, until William Burt invented the solar compass (1836). At about the same time the transit, in something like its present form, came into use.

Of course, from the beginning, the attempt was made to establish important lines as true meridians or parallels, by taking astronomical observations and finding the declination of the needle, but these observations were crude and the procedure used to prolong these


DEPARTMENT OF THE INTERIOR
BUREAU OF LAND MANAGEMENT
PRINCIPAL MERIDIANS OF THE FEDERAL SYSTEM
OF RECTANGULAR SURVEYS
1960
STATE OF OHIO ENLARGED
NOTE: The shading shows the area governed by each principal meridian and its base line
Scale in Miles
Texas
ALASKA
Arctic Circle
Copper Riv M
Seward Mer
Fairbanks M
Me
NH
Vt
Mass
Conn
R I
N Y
Penn
N J
Del
Md
Va
W Va
N C
S C
Ga
Ky
Tenn
Ellicotts L
Ellicotts Line
Ohio River Base
Muskingum R B
Scioto R B
Tallahassee M
St Stephens M
Huntsville M
Chickasaw M
Washington M
St Helena M
Louisiana Mer
Michigan Mer
4th PM
Sixth Principal Mer
Cimarron
Boise Mer
Willamette Mer
San Bernardino
Humboldt
Base Line

lines was imperfect so that even the principal meridians of the early surveys were not straight lines on the ground.

In running out lines with the compass, peep-sights were used and a foresight, determined by a compass bearing, was taken on a range

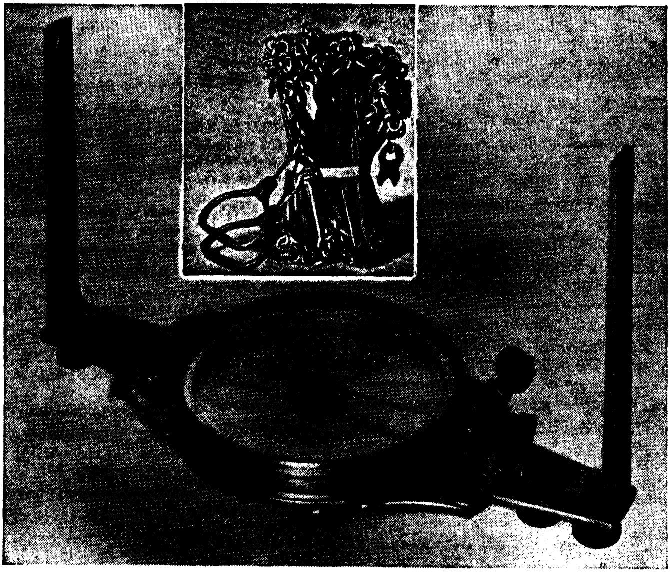


FIG. 12-2. Early Surveyor's Compass and 66-ft. Chain.

pole or some landmark ahead. The chainmen then measured the distance and the compass was moved to that point, where another foresight was taken and the work continued. Thus, no backsights were used; and, if a tree was found to be on line, the instrument was carried past it, set up by estimation on the line produced, and a new foresight taken. It is evident that any line run by this procedure would not be a straight line, but would have a small angle introduced at every instrument station.

Hence, by this procedure, no line would be established as a straight line for a distance greater than the segment between two adjacent corners; and since, from an early date, the laws prescribed quarter-section corners to be set on all lines at half-mile intervals, it may be said that for all land lines established prior to 1894, when the use of the compass was abolished, a section of land is not a square tract having four sides, but it is an irregular tract having eight sides.

There is no definite date when it may be said that the telescope came into general use, or when the use of backsights was introduced to establish straight lines. The different surveyors employed different instruments and methods, except as these were prescribed by law, or by the Surveyor General's instructions, so that at the same time, one surveyor might be using a Burt solar compass, another might be using a transit, and another might be using a peep-sight compass for doing the same kind of work.

12-5. Distances The chain used on the early surveys was two "poles," "perches," or "rods," i.e., 33 ft long, composed of 50 links. All recorded distances, however, were expressed in units of a four-pole chain, or 66 ft (see Fig. 12-2). It was made of wire, and the many links exposed many wearing surfaces so that the length of the chain increased, perhaps as much as a half-foot in a season. A means of adjusting the length of the last link was provided, and comparisons with a "standard" chain were prescribed, but the crude methods used to deal with the errors of measuring are indicated by the poor instructions which governed the field work.

The Manual of Instructions of 1902 uses the phrase "field chains or steel tapes," which is the first evidence of the use of the steel tape in the work of the government land surveys.

The use of the link chain has been discontinued for some years, and the Manual of 1930 prescribed the use of steel tapes of various lengths from 1 to 8 chains (66 to 528 ft) long to be used where field conditions are most suitable.

12-6. Conditions Affecting Early Surveys Permissible limits of error in the field measurements of the public lands were prescribed from a very early date, but at first these were vague and indefinite. Subsequent instructions became more specific, providing limits of error for the lengths and directions of the various lines run. The fact that the measurements actually made were often in error far beyond the limits prescribed arose from many conditions other than limitations of the instruments or procedure in use, some of which should be briefly mentioned:

(a) *Land Values.*—The common allotment of land to early settlers on the public lands was a "homestead" of a quarter section, 160 acres, at a cost of $1.25 per acre, including certain requirements as

to improvements to be made within a specified time. Accordingly, it was believed that high accuracy in the field measurements was not warranted by such a low land value.

(b) *The Contract System.*—From the beginning, haste was imposed principally by the condition that the surveys were made under contracts at a specified sum per mile. Different rates were paid depending on the importance of the lines (i.e., whether they were section lines or range lines, etc.) and on the character of the terrain (i.e., whether it was open, wooded, flat, hilly, or swampy). At all times, however, the financial return to the contracting surveyor depended on the speed with which the lines could be run.

(c) *Supervision.*—All contracts were executed under oath as to the completeness and accuracy of the results, but supervision and inspection was, at first, totally lacking and was inadequate until about 1880.

It may be added that unsatisfactory work often resulted from pressure for haste exerted by settlers who occupied unsurveyed land and were anxious to have their boundaries fixed; that Indians frequently were hostile; and that competent surveyors were not available in sufficient number for the vast amount of work to be done.

However, having referred to the many sources of error in the original surveys, it should be said that much of the work was surprisingly well done, and in view of the adverse conditions, those surveyors who, over wide areas, achieved such excellent results, have well earned the gratitude and respect of their successors. The U.S. Land System has proved itself, and will continue to be an inestimable benefit.

12-7. Field Notes and Plats The importance of keeping an accurate and intelligible record of the field measurements of the public land surveys was recognized from the beginning, and careful and explicit instructions have always governed this work. Of course, the notes returned did not always conform to specifications; and either because of adverse field conditions, or because of shoddy or fraudulent work of the surveyors that the lack of supervision and inspection permitted them to do, the actual present field measurements often vary widely from those shown in the notes or on the accompanying plats. However, the original record as shown in the field notes and on the plats is constantly required by present surveyors as they at-

tempt to retrace the lines established by the original surveys; and many serious mistakes are now made by inexperienced surveyors who fail to make proper use of these sources of information.

These records were returned to the Surveyor General in charge and subsequently they have been filed at the various U.S. Land Offices, where work is still in progress, or in the capitals of the states in which the lands are situated. A list of these depositories is given in the 1947 "Manual of Surveying Instructions of the Bureau of Land Management."

12-8. Early Principal Meridians and Baselines A correct understanding of the rectangular system as it was first laid out requires a statement regarding the origin of those principal meridians and baselines to which the tiers and ranges of townships are referred.

As stated above, the first subdivision of the public lands under the ordinance of 1785 was the "Old Seven Ranges," as they are called. The north boundary of this tract was established as a due west line beginning at the point where the western boundary of Pennsylvania intersected the Ohio River. It extended west for seven ranges (42 miles) and is called the "Geographer's Line," having been run by the first Chief Geographer. The range of townships was subdivided to the south until it reached the Ohio River, but it was numbered from the river to the north. Thus, this tract has no principal meridian and no baseline.

The First Principal Meridian is the western boundary of the state of Ohio extending north from the old Greenville Treaty Line. The corresponding baseline is the 41st parallel of latitude.

12-9. The 24-Mile Tracts The Manual of Instructions of 1855 includes this requirement: "On the north of the principal baseline it is proposed to have these standards (correction lines) run at distances of every four townships, or twenty-four miles, and on the south of the principal base, at distances of every five townships, or thirty miles."

Nothing is said about "guide meridians," however, until the manual of 1881, which provides for these lines to be established every four townships, thus establishing 24-mile tracts as shown in Fig. 12-3.

The manner of establishing these tracts may be described briefly as follows: The principal meridian was extended as a true north

and south line, corners being set at half-mile intervals to serve in subsequent subdivision work. The baseline was begun at the initial point, and run as a true parallel of latitude, i.e., a curved line, corners being set at half-mile intervals.

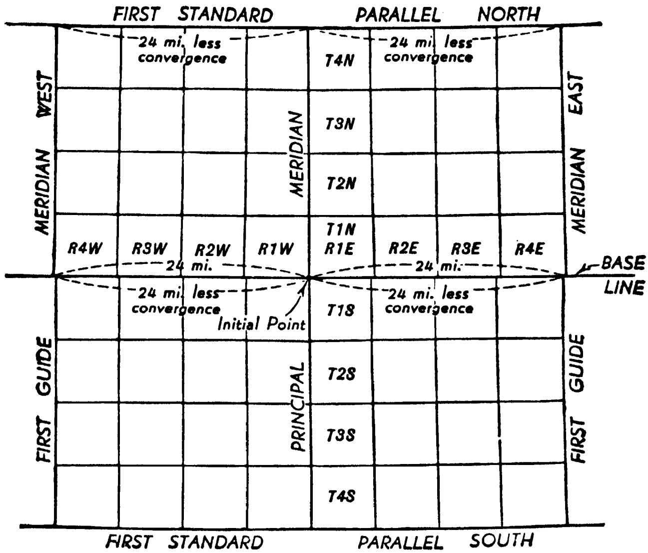


FIG. 12-3. Guide Meridians on 24-Mile Tracts

At a distance of 24 miles north of the initial point another parallel was extended east and west, called the First Standard Parallel North. Likewise at intervals of 24 miles along the principal meridian other standard parallels were run, on each of which corners were set at intervals of one-half mile.

At a distance of 24 miles west from the initial point along the baseline, a meridian was run true north, called the First Guide Meridian West. It was extended until it intersected the first standard parallel north. Because of the convergence of the meridians, this guide meridian intersected the standard parallel at a distance less than 24 miles from the principal meridian, and terminated in a closing corner. On the standard parallel, the guide meridian was extended north from the standard corner, this point being 24 miles from the principal meridian. Thus, the guide meridian was a straight line for 24 miles only and was jogged at each standard parallel. Corners were set every half mile along this line.

12-10. Township Exteriors and Subdivision of Townships The establishment of the township exteriors and the subdivision of the townships involved many details. Procedures, moreover, have changed from time to time, so that it would be impossible to describe them within this text. However, a general picture of the results is shown in Figs. 12-4 and 12-5. The most important statement to be

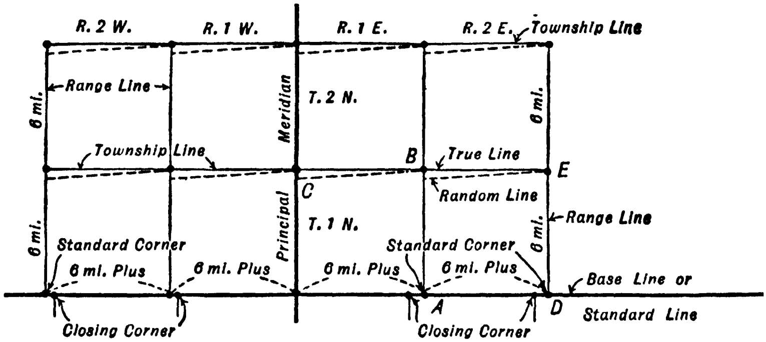


FIG. 12-4. Township Exteriors, Showing the General Procedure Used Before 1855.

made about these procedures is that, in his attempt to recover or to replace original corners, the present surveyor must attempt to "follow the footsteps" of the original surveyor. To do this he must know what the procedure was at a given time and place when the original corners and lines were established. These instructions, issued prior to 1855, have been made available by the research of Professor John S. Dodds; since that date the instructions are given in the various Manuals of Instructions of the General Land Office and the Bureau of Land Management. See references p. 271.

12-11. Subdivision of Sections It has never been the practice of the General Land Office to establish corners within the sections. Such corners and subdivisional lines were left to be run by county or private surveyors. Accordingly, various methods were used to locate the center of a section. This caused much confusion, particularly in Illinois, and at a conference of the surveyors of that state, in 1857, Abraham Lincoln was asked to render an opinion on the matter. Because of the importance of this opinion on this subject and of the legal principle involved, the complete statement submitted by Mr. Lincoln is given herewith:

The 11th Section of the Act of Congress approved Feb. 11, 1805, prescribing rules for the subdivision of a section of land within the United States system of surveys standing unrepealed, in my opinion, is binding on the respective purchasers of different parts

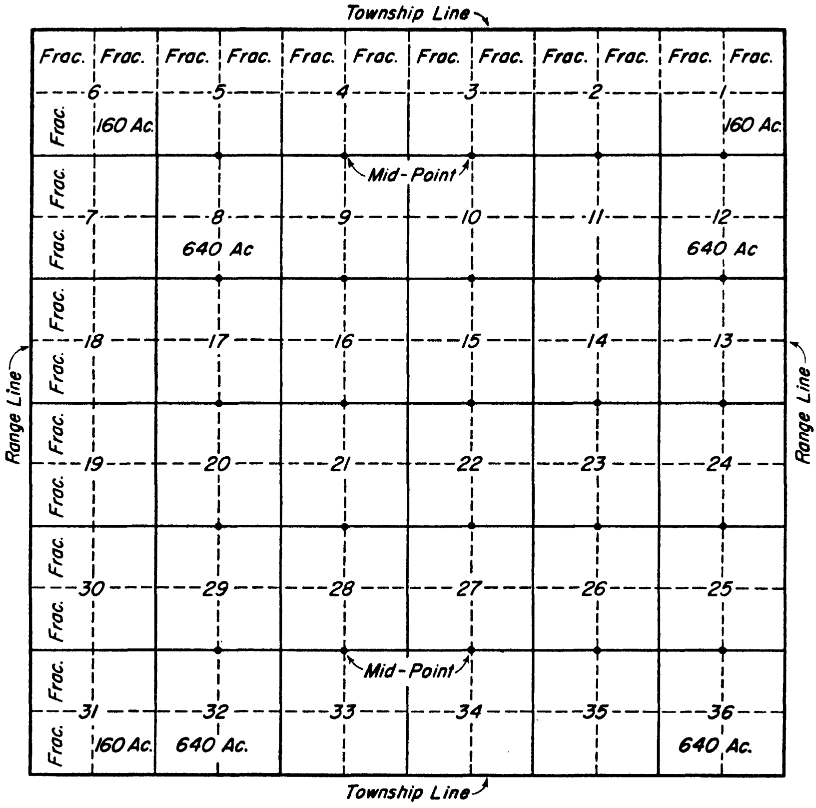


FIG. 12-5. Subdivision of a Township (Before 1855).

of the same section, and furnishes the true rule for surveyors in establishing lines between them. That law being in force at the time that each became a purchaser, becomes a condition of the purchase. And by that law I think the true rule for dividing into quarters any interior section or section which is not fractional, is to run straight lines through the section from opposite quarter corners fixing the point where such straight lines cross or intersect each other, as the middle or center of the section.

Nearly, perhaps quite, all the original surveys are to some extent erroneous, and in some of the sections quite so. In each of

the latter it is obvious that a more equitable mode of division than the above might be adopted; but as error is infinitely various perhaps no better single rule can be prescribed. At all events I think the above has been prescribed by the competent authority.

Springfield, Jaunary 6, 1859 —A. LINCOLN.

12-12. The Subdivision of Quarter Sections Following the general method of subdividing sections into quarters, the quarter sections are subdivided into quarter-quarter, or sixteenth sections by setting the quarter-quarter corners midway between the section and the quarter corners, and between the quarter corners and the center of the section. Then the center of the quarter section is established by the intersection of straight lines run between the opposite quarter-quarter corners of the quarter section—except that the sixteenth corners along the north and west boundaries of the township are established by a proportionate measurement (see Art. 12-27) of 20 chains north or west, as the case may be, from the quarter corners adjacent to these township boundaries. The fractional sixteenth sections against the north and west boundaries are numbered as lots, and the dimensions and acreages are shown on the plat.

12-13. Meander Lines The traverse of the margin of a permanent natural body of water is termed a *meander line*. Such lines were run as nearly as possible to conform to the mean high-water line. They served the purpose of providing data from which to calculate the areas of tracts of land made fractional by bodies of water. The conditions for and methods of running such lines were prescribed by the Government Land Office. In general, they follow the margins of lakes whose areas are as much as 25 acres, and of streams whose right-angle widths are 3 chains or more.

It may be noted here that, although meander lines are shown on the plats and are used to calculate the areas of land lots, they do not constitute property lines. The ownership of property bordering on bodies of water is discussed under the subject of riparian rights (see Art. 12-29).

12-14. Subdivision by Protraction Since no corners were set within the sections on the original surveys, the manner of subdividing the sections has been shown by lines protracted on the government plats. Fig. 12-6a shows these lines for usual conditions. The

fractional sixteenth sections along the north and west boundaries of the township are called lots and are numbered as shown. The dimensions and acreages for these lots are also given.

Sections which include meanderable rivers or lakes have the fractional sixteenth sections shown as lots and numbered as in Fig. 12-6, *b* and *c*.

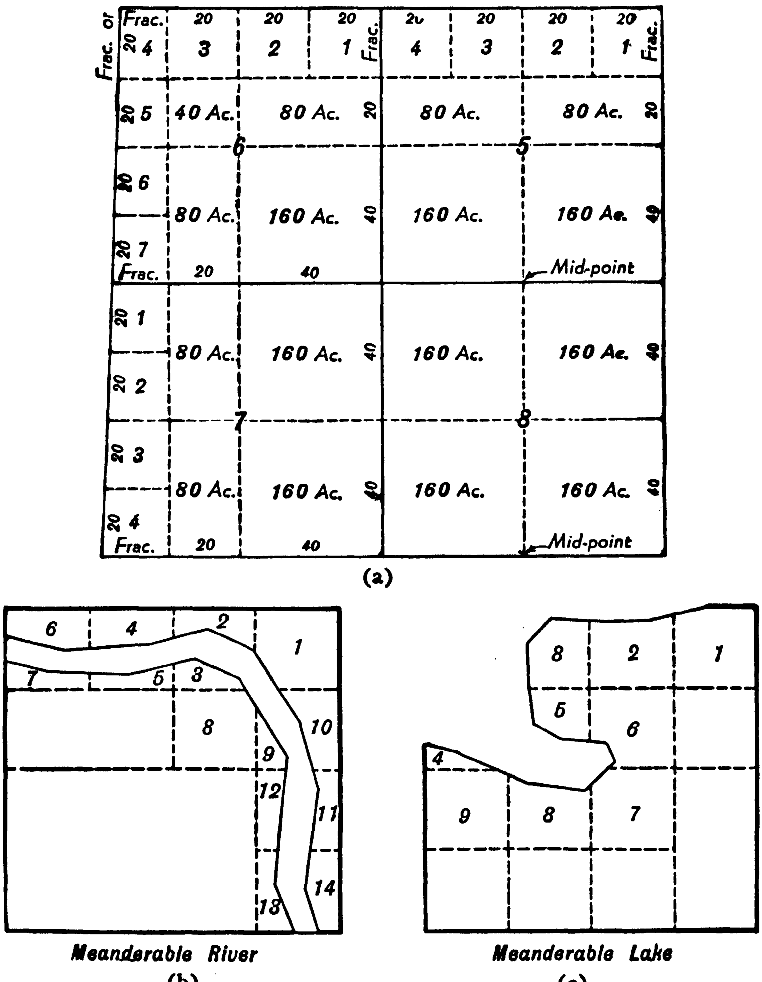


FIG. 12-6. Subdivision of Fractional Sections.

12-15. Kinds of Corners The term "corner" has two meanings: (1) it refers to the point fixed on the ground by measurement along a line from another established point, or by the intersection of established lines; and (2) it refers to the physical object or marker which serves as a more or less permanent monument at the given point. In the following paragraphs of this article the term has the first meaning.

Marking Corner.—A marking corner is the point established by the survey measurements as the actual location of one corner of a regular tract of the subdivisional system. Such corners are designated by many different terms, depending on the location of each one in the system, such as a quarter corner, section corner, township corner, closing corner, etc.

Witness Corner.—Where a corner fell in a body of water or other place where it was impracticable to fix the corner itself, a witness corner was set on each one of the survey lines leading to the inaccessible corner and at a known distance from it. Thus, the corner itself was fixed by means of the auxiliary witness corners.

Meander Corners.—Every meander line began at one, and ended at another, point on a line of the regular subdivisional system. At each of these two points a meander corner was established. Also, as the meander line was extended, if it intersected any of the regular subdivisional lines, a meander corner was established at each intersection. Of course, by coincidence, a marking corner or a witness corner might also serve as a meander corner.

12-16. Corner Materials The marking of the original section corners included such a variety of materials and methods that it is quite impossible to make any statements regarding them that will correctly describe them for more than a small territory and those set during a short period of time. Regarding these markers, the many Surveyors General issued their own instructions and these varied from time to time. However, it should be said in this connection that the original corner marker is incontestable evidence that fixes the position of a corner. Accordingly, the most satisfactory restoration of the original corners requires the surveyor to know the instructions under which these corners were established and to consult the field notes returned from the original survey.

These instructions were elaborated in later years as experience proved the need for more complete and permanent evidence of the

corner locations. Subsurface materials of stones, or broken crockery or charcoal were placed; on the prairie, mounds of earth, or mounds of stones, and pits were made in specified forms to mark the corners. In all cases, the surveyor was instructed to record in his notes the material and method used to mark each corner.

The quality and manner of marking the corners have gradually improved until the 1930 Manual of Instructions prescribes iron posts, 3 ft long, filled with concrete and having a metal cap on which the identifying letters and numerals are stamped. If the corner falls in a roadway, a suitably marked stone is buried as a subsurface mark, and a witness corner is established nearby, outside the roadway.

12-17. Corner Accessories In the attempt to establish the position of a corner as permanently as possible, other objects in the immediate vicinity were used to evidence the location of the corner itself. Such objects are called *corner accessories* and these may be such natural or artificial objects as were used for the corner materials, described above. Of course, they were given specified markings, and careful descriptions were entered in the field notes.

In timbered areas, bearing trees described above were universally used. Clearance of the land for lumber or for cultivation, forest fires, and other agencies have destroyed many of these trees, all of them in some regions, but the skillful survyor may often discover slight bits of evidence which re-establish beyond doubt the position of the original corner.

12-18. Perpetuation of Land Corners From ancient times the evidence of land ownership has consisted of physical objects on or in the ground that mark the corners or boundaries of the tract owned. That condition still prevails. A legal title deed may confer ownership of the land described in the deed, but that which determines what is really owned is the physical evidence on the ground. This condition makes it absolutely necessary for the purchaser to see the physical objects which mark his boundaries if he is to know exactly what land he is buying.

This necessity of maintaining the identity of land corners is beset with many difficulties resulting from weather, building and grading operations, and other agencies that may disturb or destroy evidence of the original location of a corner or a boundary. Within recent

years methods have been developed whereby the position of a marking corner can be permanently fixed by a tie to the national network of horizontal control established by the U.S. Coast and Geodetic Survey. A description of the procedure is beyond the scope of this book, but some of the features of this method are shown in Fig. 12-7.

The essential feature of the method is that the rectangular coordinates of a marking corner are determined with respect to the State Plane Coordinate System fixed by the U.S. Coast and Geodetic Survey. The x direction is east or west, and the y direction is north or south. These values are shown in Fig. 12-7 for each of the four corners of the area in the given example. When these values are known, then, if the marking corner is destroyed, it can be replaced by means of its known coordinates with respect to the permanent state-wide control system. This procedure is highly desirable and is gradually coming into use.

12-19. What Present Surveys Reveal In the preceding pages something has been said of the conditions under which the original surveys were made. Some of these conditions may be listed in summary here: (1) the instruments used, i.e., the compass and two-pole chain, rendered highly accurate measurements impossible; (2) the contract system placed upon the surveyor the incentive for speed rather than accuracy; (3) the lack of training on the part of many surveyors rendered them incapable of interpreting or properly applying the instructions which were intended to govern their work; (4) the lack of supervision or inspection permitted inaccurate and fraudulent surveys to stand.

An important part of the land surveyor's work now is the retracing and restoration of the original lines and corners of the government surveys. In some areas where those surveys have been made in more recent years this work can be done with considerable ease and satisfaction, but in many of the central and southern states where the original surveys are more than a hundred years old, this work is difficult and requires a thorough knowledge of the procedure used in the original work, and resourcefulness and good judgement in dealing with whatever evidence can now be found as to the location of the original corners.

As some indication of what the surveyor may expect to find, a few examples of conditions revealed by present surveys are given below.

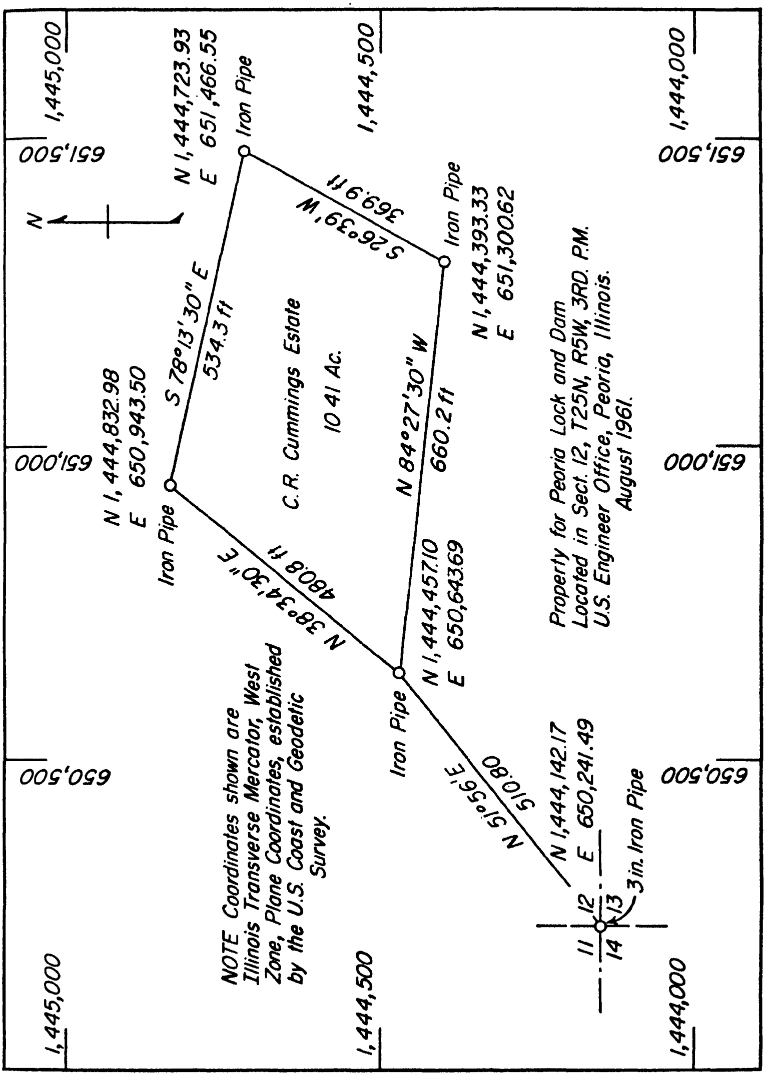


FIG. 12-7. Property Corners Fixed by State Plane Coordinates.

These are extreme examples, but they will serve to emphasize the fact that the present surveyor must expect to find that (1) no present measurement will agree with that shown on an original plat; (2) undetected mistakes are likely to be present anywhere in the original measurements and in the recorded notes; (3) no line can be regarded as being straight for more than one-half mile; (4) while all quarter corners, except those in the north and west sections of a township, are intended to be equidistant between adjacent section corners, such is frequently not the case.

Case I. Fig. 12-8a shows two plats of the same section, the first as returned by the deputy surveyor in 1884, and the second as returned by a recent survey by the General Land Office.

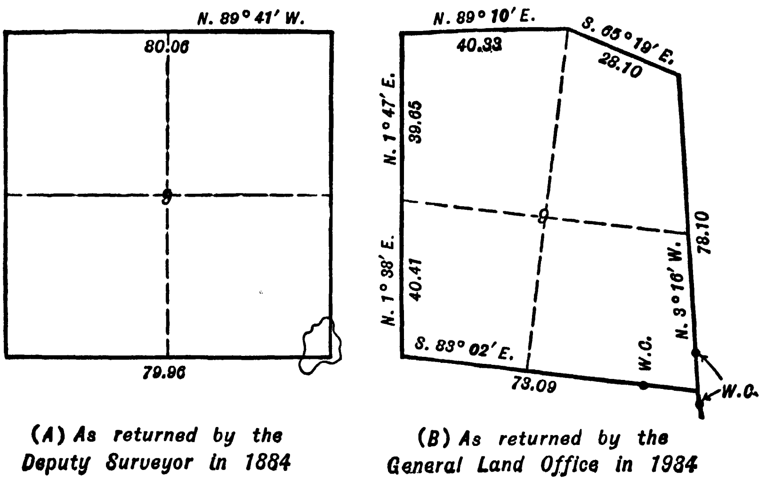


FIG. 12-8a. Two Plats of Sec. 9, T. 64N., R. 1W., 4th P. M. in Minnesota.

Case II. Fig. 12-8b shows a quarter corner in Wisconsin misplaced 10 chains west of its correct position. However, the original quarter corner was found in place with two bearing trees to witness it, so that the owner of section 27 concluded he could not contest the line in court.

II. Legal Aspects

12-20. Legal Authority of the Surveyor The surveyor should understand clearly that, in the event of a dispute as regards the location of a corner or propery line, he has no judicial authority. If he

is a competent surveyor he will gather the evidence and interpret it in such a manner as usually to bring the parties to agreement out of court. But, if no agreement can thus be reached, the surveyor has no authority to impose a settlement. This is strictly the prerogative of

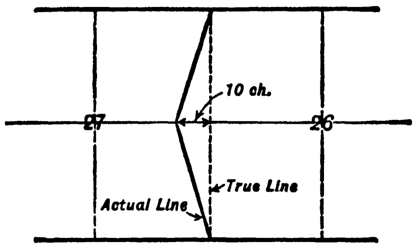


FIG. 12-8b. Quarter Corner Misplaced 10 Chains West of the Correct Location.

the court, and the surveyor serves only as an expert witness in the case.

Also, to avoid unnecessary litigation and to prevent mistakes in present surveys arising from ignorance of the legal principles involved, the surveyor is required to be well informed in the laws relating to land surveys within the state and jurisdiction where the property is located. This fact is evidenced by the preponderant weight in examinations for the surveyor's license which is given to the legal aspects of surveys as compared with the physical measurements required.

12-21. General Rules Governing Surveys of Public Lands The following list of general rules, which are the result of congressional legislation, is taken from the Manual of Instructions (1930).

First. That the boundaries and subdivisions of the public lands as surveyed under approved instructions by the duly appointed engineers, the physical evidence of which survey consists of monuments established upon the ground, and the record evidence of which consists of field notes and plats, duly approved by the authorities constituted by law, are unchangeable after passing of the title by the United States.

Second. That the physical evidence of the original township, quarter section, and other monuments must stand as the true corners of the subdivisions which they were intended to represent, and will be given controlling preference over the recorded directions and lengths of lines.

Third. The quarter-quarter section corners not established in the process of the original survey shall be on the line connecting the section and the quarter-section corners, and mid-way between them, except on the last half-mile of section lines closing on the north and west boundaries of the township, or on other lines between fractional or irregular sections.

Fourth. That the center lines of a regular section are to be straight, running from the quarter-section corner on one boundary of the section to the corresponding corner on the opposite section line.

Fifth. That in a fractional section where no opposite corresponding quarter section corner has been or can be established the center line of such section must run from the proper quarter-section corner as nearly in a cardinal direction to the meander line, reservation, or other boundary of such fractional section, as due parallelism with section lines will permit.

Sixth. That lost or obliterated corners of the approved survey must be restored to their original locations whenever it is possible to do so. Actions or decisions by surveyors, Federal, State, or local, which may involve the possibility of changes in the established boundaries of patented lands, are subject to review by the State courts upon suit advancing that issue.

12-22. Deed Descriptions The purpose of a deed to real property is to convey ownership of a tract of land from one party to another and to evidence title in the owner. The purpose of the description which must be a part of the deed is to furnish the information which is necessary to identify the boundaries of that particular tract on the ground, both at the time the conveyance is made and at any future time. It is evident, therefore, that the proper composition of a deed description requires a knowledge both of the law and of surveying, and something of these requirements is indicated in the following paragraphs.

12-23. Kinds of Descriptions Land descriptions may be grouped into four general kinds as follows: description by reference to (1) natural objects and adjoiners, without numerical data, (2) metes and bounds, (3) the public lands system, and (4) urban subdivisions.

This classification is not a definite one, for many descriptions will include characteristics of two or more of these classes, but it will serve the purpose of indicating the principal features of the various kinds of descriptions.

The first class includes those descriptions which refer to such natural objects as a tree, the center line of a road, the thread of a

stream, etc., or a boundary line may be described by giving no information except the names of the adjoining owners.

Such descriptions may contain some numerical data which may aid in fixing the boundaries, but in descriptions so imperfectly drawn, the numerical data are likely to be so faulty as to be worthless. No reputable surveyor would return a description of this kind now, but in early times, when land values were small, such descriptions were thought sufficient. The following is an example:

> Beginning at a stone in the highway leading from Portsmouth to Springfield, and on the east fence line of land of Levi Brown; thence east by the highway to land of James Green; thence north by land of Green to Spring Creek; thence westerly along Spring Creek to the east fence of land of Levi Brown, thence by land of Brown to the point of beginning; containing 22.40 acres, more or less.

The second class includes all pieces of land not of the first class and not regular tracts of the public lands system or an urban subdivision. The description begins by carefully describing the point of beginning and then giving the distance and bearing of each course from one corner to the next, in consecutive order around the tract. The marker at each corner is also described. These are called "metes and bounds" descriptions, of which the following is an example:

> Beginning at a point, marked by an iron pin, 584.10 feet North of the S.E. corner of the N.W. $\frac{1}{4}$ of the S.E. $\frac{1}{4}$ of Sec. 2, T. 19 N., R. 8 E., 3rd P.M.; thence North 607.86 feet to the center line of Bloomington Road, marked by a cross cut in the pavement; thence in a Northwesterly direction along the center line of said Bloomington Road, 422.41 feet to another cross cut in the pavement; thence South 817.47 feet to an iron pin; thence East 366.74 feet to the point of beginning; containing 6.00 acres.

The third class includes all regular tracts within the areas covered by the U.S. public lands system. Thus, a forty-acre tract may be described as: "N.E.$\frac{1}{4}$, of N.W.$\frac{1}{4}$ of Section 9, Township 64 North, Range 1 West of the Fourth Principal Meridian."

The fourth class includes all regular parts of urban subdivisions. Thus, a tract may be described as "Lot No. 9, Block 4, in Hills and Dales Addition to the Town, now City of West Lafayette, Indiana."

12-24. Requirements of a Valid Description The essential requirements of a valid land description are that it shall be clear, accurate, and brief.

The great advantage of the United States public lands system is that it makes possible the use of descriptions so excellent in these essentials. The identification on the ground of a given tract may not be a simple matter, but that condition does not detract from the excellence of the system insofar as the description of any regular tract is concerned. And the same advantages apply to regular lots of urban subdivisions. The land surveying difficulties which apply to such areas either rural or urban, arise from other sources, which are discussed elsewhere, but the descriptions of these tracts can hardly be improved.

Faulty land descriptions are likely to occur where any irregular tract must be described by "metes and bounds," and some of the considerations which apply to such descriptions will be discussed in the following paragraphs.

Clarity.—A land description should be so clear that it is subject to but one, and that the correct, interpretation; this condition should apply not only at the time the description is written, but at any future time. Therefore, the writer must keep the point of view that his description must, if followed explicitly by the surveyor, mark out the correct boundary on the ground either now or at any later time.

So much confusion has arisen from the lack of clarity in deed descriptions that many legal principles have become established as a necessary means of giving the correct interpretation to faulty descriptions. Some of these principles are discussed in the next article.

A first requirement to this end is the accurate use of words and phrases. For example, in giving directions or bearings, the word "north" should be used to indicate due north only, i.e., the direction parallel with the reference meridian of the survey; whereas, "northerly" may indicate any direction in the first or fourth quadrants. Also, the terms, "at right angles to" or "parallel with" should refer only to lines between which the angles are 90° or 0°, respectively. Other examples might be given, but it is evident that only confusion can result from the incorrect or inexact use of words or phrases.

Another source of confusion and ambiguity in land descriptions results from the incorrect use of punctuation marks. Of course, these are often misplaced in copying and recording, but great care should be taken in the sentence structure and phrases used so that the cor-

rect meaning will depend as little as possible upon the aid of punctuation marks.

A correct plat is always helpful and sometimes necessary for clarity in descriptions. Hence, it is desirable that a correct plat should accompany any resurvey, and reference made to it in the description, whereby it becomes a part of the deed (see Fig. 12-13).

In order to fix the location of a tract, it is frequently necessary to refer to some established monument, road, or street line in the near vicinity. Care must be taken that such monument or line is definite, and identifiable upon the ground, and that the "point of beginning" of the survey shall refer to a corner of the tract surveyed and not the monument to which the tract as a whole is referred.

Accuracy.—A correct survey should, of course, precede the writing of any metes and bounds description. Any attempt to scale dimensions shown on such a plat, for the purpose of writing a present description, is sure to result in error and confusion.

For example, assume the following conditions: a city lot is shown on a plat as being 100 ft long and 50 ft wide, its length being in a north and south direction; a previous deed has conveyed out the west 12 ft of this lot; and the actual width of the lot on the ground is 49 ft.

The plat indicates the remainder of the lot as being 38 ft wide, but, as a matter of fact, it is only 37 ft wide. It is evident that it would be incorrect for the lawyer, or grantor, to describe this remainder as the "east 38 ft of said lot" because by the previous deed the remainder is only 37 ft wide, and since a deed cannot convey that which is not owned; therefore with respect to the 1-ft strip in question, the present deed is impotent. This remaining tract should properly be described as "the whole of said lot, except the west 12 ft."

This example also shows the necessity for a present survey, if the grantee is to be saved the disappointment of subsequently finding that his lot is only 37 ft wide instead of the platted 38 ft.

Surely, the grantee should be as greatly concerned that the description of his tract is an accurate one as he is that his legal title is a clear one. A lawyer understands the necessity for a valid abstract of title, but he frequently does not understand the condition that a valid description must depend on an accurate field survey.

The distances and directions of a metes and bounds description should be accurate and complete so that the error of closure of the

boundary can be computed. This condition will insure that no essential data have been omitted and will indicate the precision of the survey, as regards the accidental errors involved. It should in no way depend upon the data contained in the descriptions of adjoining tracts.

Brevity. Brevity is essential in land descriptions because brevity enhances clarity, and it reduces expense and mistakes in subsequent records. It is for this reason that a plat is essential as a part of a description, for it conveys in small space and in legible form information that otherwise would require many words.

It is common practice to use both numerals and words in recording distances and directions; as for example, "north twenty-two degrees and fifteen minutes west (N 22°15′ W), a distance of four hundred seventy-five and thirty-two hundredths ft (475.32 ft)." The words add nothing to clarity, and increase the chances for mistakes. This course should be described simply as "N 22°15′ W; 475.32 ft." Other examples of wordy descriptions could be given, but the general rule applies to each one—that it should be stated simply and directly.

In summary, it may be repeated that the final test of any description is whether or not the particular tract described can be satisfactorily identified on the ground.

12-25. Interpretation of Deed Descriptions In making a land survey the surveyor must refer to the deed description, and in many cases the description is faulty because of omissions of essential data or because of conflicting calls. Court decisions have established a few general rules which should govern the surveyor in making his interpretation.

1. The best interpretation is that which most plainly and completely gives effect to the intentions of the parties to the deed, as revealed by all of the evidence available.

2. As regards conflicts between the calls of a description, the order of precedence is as follows: (a) a natural corner or boundary, as a tree or a stream, will stand as against an artificial boundary, as a stake, stone, or a fence; (b) an artificial corner or boundary, if it can be identified, will stand against other conflicting calls as to direction, distance, or area; (c) the corner or line of an adjoiner, if it can be definitely identified, will control over calls for direction or distance or area; (d) in case there is a conflict between the boundary

dimensions and the calculated area, the former will prevail over the latter, assuming of course that the boundary dimensions are consistent with the evidence as to the corner monuments.

3. If a description is faulty by reason of any obvious errors, or omission of essential data, the attempt is made to render it valid rather than void. Thus, if a dimension is incorrect by a full tape length, or if the length or bearing of one side of a field has been omitted, and if otherwise the description and evidence on the ground are satisfactory, the obvious omission or mistake will be corrected and the deed will be held valid.

4. If two interpretations are possible, the one that favors the purchaser will be taken.

12-26. Excess and Deficiency Every land survey, either urban or rural, requires the retracement and the remeasurement of property lines and, because of the inherent errors in all observations and possible mistakes, the recent measurements never agree exactly with the original; therefore, there is always some excess or deficiency to be adjusted. The discrepancy between a recent and a previous measurement may be small and the adjustment a simple one, but frequently the discrepancy is considerable, and the adjustment complex, especially where the line has been divided into a number of segments, so that the aid of established legal principles is helpful.

The general rule is well stated by the Supreme Court of the State of Nebraska as follows:

> On the line of the same survey, and between remote corners, the whole length of which is found to be variant from the length called for, it is not to be presumed that the variance was caused from a defective survey of any part, but it must be presumed, in the absence of circumstances showing the contrary, that it arose from imperfect measurement of the whole line, and such variance must be distributed between the several subdivisions of the line in proportion to their respective lengths.

This rule has a common application where urban property is subdivided into lots and blocks, with dimensions shown on a plat and where a given parcel is described by its lot number.

The rule applies to all of the lots in a given block, even though one at the end may have a frontage dimension different from all of the others. It is sometimes supposed that such a lot, being irregular, should be considered a remnant and be assigned all of the excess or

deficiency found within the block. But, unless the dimension of such a lot is omitted, or other modifying conditions apply, it is held that any discrepancy shall be apportioned among all of the lots within the block.

The rule applies also where a tract described as containing a given acreage is subdivided by stating the number of acres in each subdivided tract. Thus, if an owner wills his farm of 160 acres to his two sons, giving to one the north 60 acres and to the other the south 100 acres, and if a survey shows the whole tract to contain 180 acres, it will be divided between the sons in the proportion of 60 to 100, each receiving his proportionate share of the discrepancy.

The rule also applies commonly to the relocation of lost corners in the U.S. rectangular system, where proportionate measurements are to be made between the nearest adjacent existing corners (see Art. 12-28).

The general rule that any excess or deficiency shall be apportioned throughout the tract in which it occurs does not apply in some cases. Four such cases are stated below.

1. If all of the lots in a block are given dimensions except one. For such a case it is plainly evident that the irregular lot was intended to include whatever remnant was left.
2. If streets intervene between the blocks of a subdivision and if the streets are monumented and have been used for some time, any discrepancy found in one block may not be apportioned among other blocks across street lines, because street lines used by the public soon become fixed.
3. If a tract of land has been partitioned by separate deeds by metes and bounds at different times. In this case, the tract last conveyed will be apportioned any excess or deficiency that may be found.
4. If erection of buildings and other acts of occupation show that ownership has been claimed according to platted dimensions.

12-27. The Search for Obliterated Corners A distinction is made between *lost* and *obliterated* corners. An obliterated corner is one which is not immediately apparent, but for which sufficient evidence in the immediate vicinity is available to re-establish the corner. A lost corner is one for which the evidence in the immediate vicinity is insufficient and for which a resurvey is necessary.

At the present time, every survey requires the location, if possible, of existing corners. This may be a simple matter if the work of the previous surveyor was well done and if the lapse of time has not been great. Unfortunately, both of these conditions are frequently adverse, and it requires careful and thoughtful procedure to establish the position of a corner for which much of the original evidence is missing.

The general principles which govern the restoration of obliterated or lost corners are stated explicitly in the first, second, and sixth of the general rules governing the resurveys of the public lands as provided by the various acts of Congress, Art. 12-21.

In applying these general principles to a specific case the surveyor is required: (1) to know the procedure used in the original surveys, (2) to provide himself with all data, i.e., original survey notes and plats, and the records of more recent surveys in the vicinity of the obliterated corner, and (3) to exercise good judgment and discernment in discovering and evaluating all possible evidence pertaining to the obliterated corner.

The kinds of evidence that may be used to restore a corner in rural regions include: (a) the corner itself, (b) accessories, (c) fences, (d) roads, and (e) living witnesses.

Of course the corner itself is the best evidence, and every reasonable effort should be made to find it. If it was a wooden stake, care must be taken that this evidence be not destroyed or overlooked. Having determined by preliminary measurements, as nearly as may be, the probable location of the corner, the search is begun cautiously by slicing off the surface material. If the stake has rotted, its position may be indicated by the rotted wood or discolored soil; or, sometimes in firm soil, the hole formerly occupied by the stake will be found plainly marked. If either of these evidences can be found, the position of the corner is as well recovered as though the stake itself were in place. If it appears that the soil has filled-in over the corner, this condition is usually made apparent by the different quality and color of the filled material. If the corner object is a stone, an excellent tool with which to search is a rod, about $\frac{1}{4}$ in. in diameter, 3 or 4 ft long, having an oval bead or ferrel, slightly larger than the rod, fitted over one end and a suitable handle at the other end. This rod can be pushed down vertically through 3 or 4 ft of nearly any kind of soil, including a packed roadway, to find a buried stone. By systematic procedure, a considerable area can be investigated by

this means in a short time, much more efficiently than can be done with a spade.

The records of any bearing trees or other reference objects should be found in the notes of the original survey or on the plats of other previous surveys. In this search, the descriptions of adjacent property should be investigated, since these might contain indispensable information. If any bearing trees or other accessories can be found, often the corner can be restored as satisfactorily as though the corner mark itself had been found.

Fences offer good evidence as to the location of a corner, since it is probable that the fence was built on the line when the corners were apparent. However, fences must be looked on with suspicion, for, in early times, a mutual agreement was frequently made whereby farmer A would build a line fence over a rod or two on B's land, and thus secure the use of this strip of land in return for the expense of building the fence. After the lapse of years, the parties who had knowledge of this arrangement may have died or moved away and their successors left in ignorance of the true location of the property line. Or, the fence may not have been a property line at the time it was built and no attempt was made to place it on the line. However, if no such evidence of this kind can be found, it is a reasonable assumption that a fence marks the original land line.

A corner which is known to have been located in the center of a road will naturally be looked for on the centerline between existing fences. But again, the history of the fence lines must be carefully investigated to determine whether or not they have been moved since the corner was placed. Frequently when a road is widened, the change will be made on one side only, and hence the centerline between fences no longer marks the original property line.

Living witnesses frequently can give valuable information as to the location of an obliterated corner. It often happens that men living in the vicinity were present when the corner was set, or have seen it on subsequent occasions, and can recall accurately its exact location. There is a wide difference, however, in the ability of persons to remember the locality of a place they have previously seen, and considerable allowance must be made for this fact. In any case, such evidence must be supported by other evidence, but it may be of great assistance in relocating a corner.

The kinds of evidence used to search for a corner within city lim-

its are much the same as those in rural regions, although conditions are quite different. Building operations and street improvements effect many changes that destroy evidence as to the location of property lines; for this reason every engineer should be most careful to set and reference any corner so as to render it as permanent as possible. Lot corners are likely to be covered with filled-in material that is usually of different quality and color than the original subsoil. Fences are seldom on the lot lines and frequently are not parallel therewith; also, pavements and sidewalks are often not parallel with adjacent property lines. All these and many other conditions make it necessary that the work be done by a competent and experienced surveyor if satisfactory results are to be obtained.

12-28. Restoration of Lost Corners If a corner cannot be restored by any of the evidence discussed in the previous article, it becomes necessary to make a resurvey. This procedure will be discussed for the two conditions (a) when the boundary is described by metes and bounds, and (b) when the boundary is a part of the U.S. land system.

Restoring Corners Described by Metes and Bounds.—It is assumed that deed descriptions are at hand, both of the property itself and of adjacent properties; also, that one or more corners are in place. Beginning with this evidence, the surveyor retraces the existing boundaries, as nearly as may be, measuring the distances, angles, and bearings. These measurements, when compared with those of the deed descriptions or plats, will not agree exactly but will provide a proportion which, it is reasonable to suppose, will apply to the remaining boundaries to be restored. Likewise, a comparison between present and old bearings will indicate a variation which will serve as a guide in establishing the directions of the boundary courses. By the use of these proportionate measurements the entire boundary is retraced, setting temporary corners as the work proceeds. If the location of these temporary corners seems to be consistent with all evidence at hand, the corners are then established as the permanent corners. But it is possible that much conflicting evidence will remain after the temporary corners are set, and this is to be harmonized by a further study of all the evidence, and by secondary proportionate measurements. In conformance with these adjustments, the final corners are set and carefully referenced. A traverse is then run measur-

ing all the distances and angles, from which the error of closure and the area are calculated. A new description of the boundary is written and a plat prepared to complete the survey.

Restoring Corners Described by the U.S. Land System.—When it becomes necessary to restore a corner that is a part of the U.S. land system and that cannot be found by the means described in the previous article, a resurvey is made to replace the corner as nearly as may be in its original position. This procedure requires a thorough understanding of the manner in which the original surveys were executed, and it would be impossible here to describe in detail the many conditions which may be encountered. Because of the complications that may arise, the engineer is cautioned about attempting this work without full information regarding the original surveys. However, the procedure for one or two simple cases will be given to indicate the general principles involved.

1. *Single Proportionate Measurement.* Suppose that in Fig. 12-6*a* the quarter corner on the south boundary of section 7 is lost and the westerly fractional distance of the original survey was 38.42 chains; also that the two section corners are in place, and that the present measurement between them is found to be 5183.6 ft. The original measurement was 5175.7 ft (78.42 chains), and the lost corner was originally placed at a recorded distance of 2640.0 ft from the east section corner. Let the west, center, and east corners be represented by letters A, B, and C, respectively. Then the relations to be established are as follows:

$$\frac{BC\text{ (new)}}{AC\text{ (new)}} = \frac{BC\text{ (old)}}{AC\text{ (old)}} \quad \text{or,} \quad BC\text{ (new)} = \frac{2640.0}{5175.7} \times 5183.6$$

$$= 2644.3\text{ ft}$$

Accordingly, quarter corner B is to be restored by the measurement of 2644.3 ft from east corner C. This is called a single proportional measurement.

2. *Double Proportionate Measurement.* Let it be supposed that the section corner between sections 15, 16, 21, and 22, as illustrated at O, Fig. 12-9, is lost, and that the corners at A, B, C, and D are in place. Then, for the purpose of illustration, it may be supposed that the four quarter corners on the interior section lines are lost. Also, assume that the original measurements are as shown and that the present measurements of distances AB and CD are 10545.1 ft and

10571.8 ft, respectively. As above, the relations between the new and old measurements are:

$$\frac{AO\text{ (new)}}{AB\text{ (new)}} = \frac{AO\text{ (old)}}{AB\text{ (old)}} \quad \text{or} \quad AO\text{ (new)} = \frac{5282.6}{10{,}536.2} \times 10{,}545.1$$
$$= 5287.1\text{ ft}$$

Let a temporary stake O' be set at this point.

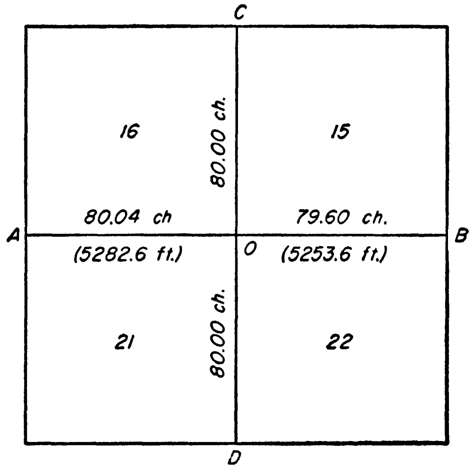


FIG. 12-9. Double Proportionate Measurement.

The original position of O was midway between C and D and, hence, it should be restored in the same relationship, and the distance $CO = 10571.8/2 = 5285.9$ ft. Let a temporary stake O'' be set at this point. It is evident that the temporary stake O' marks the longitude of the corner, and O'' marks its latitude. Hence, the corner is restored at the intersection of a north-south line passed through O' and an east-west line passed through O''. This procedure is termed double proportionate measurement.

In establishing the lines of the original government surveys it was required that the precision of the measurements should be greater on the controlling lines of the larger units than for the lines which subdivided those units. This higher precision not only was specified in the statutes under which the work was executed, but was recognized in the pay which the surveyor received. Thus, a higher price per mile was paid for establishing a baseline than for a township

line; also, a higher rate was paid for a range line than for a section line. Accordingly, it is required in restoring corners that lines which were intended to be more precise than others shall have greater consideration and effect.

For example, suppose the southeast corner of section 12, Fig. 12-5, is to be restored. This corner lies on a range line which was intended to be a straight line for 6 miles and was required to be a more accurate line than the connecting section lines. Hence, to restore this corner, single proportionate measurements are taken between the nearest recoverable corners along the range line; and any conflicting evidence contributed by adjacent section lines will have no effect.

Likewise, a corner on a township line, the northwest corner of section 3, for example, will be replaced by single proportionate measurment along the township line, and any conflicting evidence contributed by adjacent section lines will have no effect.

But the northeast corner of section 1 is a corner common to four intersecting lines of equal precision, and, hence, this corner will be restored by double proportionate measurements from the nearest recoverable corners in all four directions. Similarly, an interior section corner, as the northwest corner of section 15, will be restored by double proportionate measurements from the nearest corners in all four directions.

It is a general principle, therefore, in restoring corners, that if the lines which meet at a given corner are of different degrees of precision, single proportionate measurements are used, but, if the lines are all of the same importance, double proportionate measurements are used.

12-29. Riparian Rights Riparian rights are those vested in an owner by the condition that his property borders on the bank or shore of a stream or body of water. Such rights are often of great value as when they endow owners with many shore privileges including the right to construct docks and wharves. Because of these values occurring from riparian ownership and because of the irregular and changeable nature of such boundary lines, it is important that the surveyor be well informed concerning the laws and customs which govern the establishment and maintenance of these lines within the state in which his survey is located.

12-30. Limits of Riparian Boundaries *Nonnavigable Streams.*—In dealing with streams as boundaries, the procedure depends frequently upon whether or not the stream is "navigable." In early instructions of the U.S. Land Office regarding the running of meander lines it was prescribed that a river was to be considered as "navigable" if the water surface had a width of three chains. Obviously, the width of the water surface for a given stream varies widely with the stage of flow; accordingly, it is quite impossible in many situations to say whether a stream is to be regarded as navigable or nonnavigable. Perhaps there is no better interpretation at present than to regard a stream as navigable if it was meandered in the original survey, and as nonnavigable if it was not meandered.

Where a nonnavigable stream serves as a boundary line, it is common law that ownership extends to the thread, or center, of the stream, but the courts differ somewhat in defining the term "thread of stream." This is sometimes defined as the line midway between the usual shore lines, regardless of the position of the main channel, but the more common interpretation defines it as the center of the main channel.

The riparian rights of an owner of property along a navigable river are fixed largely by the statutes of the state in which the property is situated. In regard to the extent of such boundary lines, laws vary from state to state between three limitations, namely: (1) to the high-water mark, or bank, (2) to the low-water mark, or (3) to the center of the stream.

The "high-water mark" or "bank" may be defined as the line where "the presence and action of water is so common and usual, and so long continued in all ordinary years, as to mark upon the soil of the bed a character distinct from that of the banks."

The low-water mark may be defined as the line to which a river usually recedes at its lowest stage, unaffected by drought.

It should be added that meander lines of the public lands system are not property lines, and, unless they are specifically designated as such in a deed description, they do not limit riparian boundaries.

Ponds and Lakes.—The limits of property bordering on a pond or lake are fixed by the laws of the state in which it is situated. In a few states, ownership is limited by the shore line, but in most states, if a pond or a lake is nonnavigable, title to property bordering on it extends to the center of it. If the land tract includes all of the pond

or lake, the bed is included in the title. However, if a metes and bounds description uses the expression "along the east shore of said pond" or "thence by the edge of the lake," ownership extends to the shore only.

In Massachusetts and Maine, according to a colonial ordinance of 1647, every lake of ten (or more) acres is public.

If a lake is navigable, title of riparian owners extends to the shore, or to the center, according to the laws of the state in which it is located.

Tidewater Shore Lines.—According to common law, title to property bounded by tidal water extends to the high-water mark only, and the beach or shore between the high- and low-water marks belongs to the state. In Massachusetts, Maine, and New Hampshire, the common law was modified by early colonial ordinances, and at the present time ownership of property along tidal water in these states is fixed by statute to include the shore, if the distance between the high- and low-water marks is 100 rods or less. If this distance is more than 100 rods, ownership is limited to that distance.

12-31. Alluvium, Reliction, and Accretion *Alluvium* may be defined as the change in the location of a shore line by the gradual and imperceptible deposition of soil so as to increase the area of the contiguous land.

Reliction refers to the gradual and imperceptible recession of a shore line due either to the rising of the shore or to the subsidence, or drying up, of the body of water.

Accretion results in either of the above cases, and it is the general rule that accretion so gained is lawful and the boundary of property so affected will change with the movement of the shore line.

Such accretion, however, has its adverse counterpart. If gradual and imperceptible erosion takes place, the riparian owner will suffer a loss by the change in the shore line. Likewise, if the shore subsides, or the body of water rises, gradually and imperceptibly, the contiguous property will be inundated and the owner's acreage will thereby be diminished.

12-32. Avulsion *Avulsion refers* to the sudden and perceptible change in a shore line, due usually to a river changing its course during a flood or freshet. It is the general rule that avulsion effects no changes in property lines. Thus, if a river changes its course, or if its

banks are suddenly eroded during a flood, property lines will not be disturbed thereby. This rule applies to river boundaries between states as well as to private property lines.

12-33. The Extension of Riparian Boundary Lines The surveyor is frequently required to extend the land lines of riparian owners in accordance with their rights. This problem may be simple or complex, depending upon conditions, and calls for good judgment and an understanding of the procedures which have been established in similar cases.

(a) *Rivers.*—Where property rights extend to the center of a river, the general rule is to prolong each boundary line to the high-water mark or bank and then extend it in a direction perpendicular to the centerline of the river. Thus the west boundary of Lot 1, Fig. 12-10,

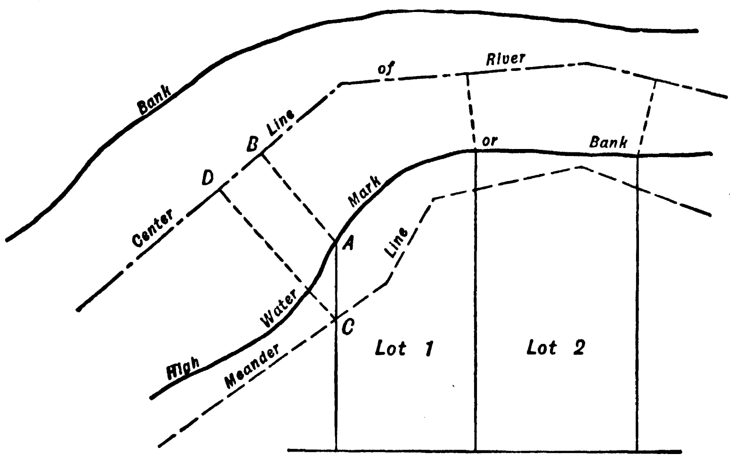


FIG. 12-10. Extension of Property Lines at a River.

is prolonged to A and then extended to B, direction AB being perpendicular to the centerline of the river.

In the public lands system when a bank has been meandered, it is sometimes held that the boundary line should be extended from its intersection with the meander line to the centerline of the river, as from C to D, but most court decisions are contrary to this view and specify the bank as the proper place at which to change the direction of the boundary line.

(b) *Accretion.*—Where boundary lines are to be extended to in-

clude an area added by accretion, many complex situations arise, but a general rule is indicated by Fig. 12-11, where A to F represents the old shore line, and A' to F' the new line to be apportioned to lots 7 to 11. This is done by dividing the new shore line so that each new

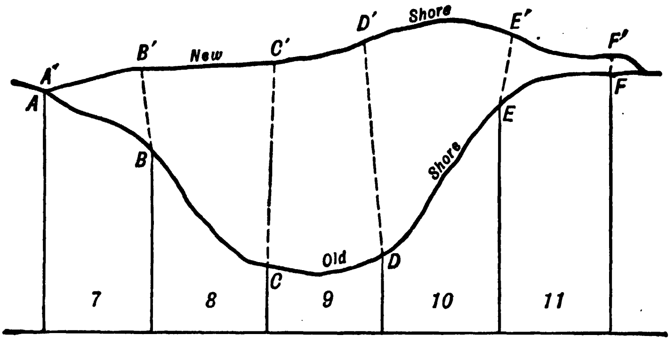


FIG. 12-11. Lines to Partition Accretion.

lot is to the whole new shore line as each old lot line is to the whole old shore line. Thus, distances AB, BC, CD, DE, and EF are measured, and the distance A' to F' is found. Then $A'B'$ is to $A'F'$ as AB is to AF; parts $B'C'$, $C'D'$, $D'E'$ and $E'F'$ are determined in a similar manner.

(c) *Tidewater Shore Lines.*—The low- and high-water lines of tidewater shores often present the same problem to the surveyor in fixing the limits for dock and shore privileges, and the same general rule is followed as indicated above for accretion. But if a bay or cove has deep indentations and sharp headlands, the low-water line to be proportioned will be taken from one headland to the other across the mouth of the bay or cove.

(d) *Lakes.*—When property rights include the bed of a lake and where the shape of the lake provides a fairly definite axis, or center line, the boundary lines are extended from the shore perpendicularly to the centerline. Fig. 12-12. But if the lake is circular in shape or has a round area as in the figure, then the boundary lines are extended from the shore to the center of the rounded part. The courts have sometimes called this the "pie-cutting" method of subdivision.

12-34. Adverse Possession *Adverse possession* is a legal term which applies to the condition where the property line between ad-

jacent owners and the titles of the adjacent properties are fixed by occupation and use of the land, as opposed to the descriptions of the properties in the deeds of ownership. Thus, it may be supposed that *A* and *B* are owners of adjacent tracts of land where a common

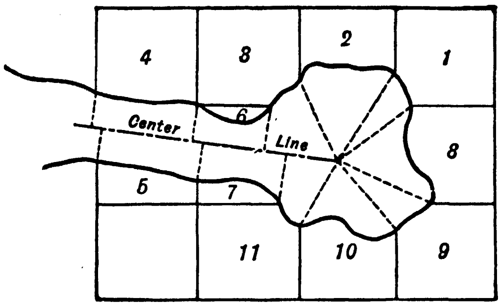


FIG. 12-12. Extension of Riparian Lot Lines for a Lake.

boundary line has the same description in each of the deeds of titles vested in both *A* and *B*. But when a fence was built, supposedly along this line, it was built over on *B*'s land, say 10 ft. Then if *A* has occupied all of the land up to the fence, believing it was his own, for the statutary terms of years (20 years in Illinois), and meeting other necessary conditions, he has acquired title to the land thus affected; and the fence, erroneously placed, has become the true boundary. This means of transfer, by occupation, of ownership and title from *B* to *A* is termed *adverse possession.*

Conditions under which title is acquired by adverse possession are as follows: occupation must be (a) actual, (b) open and notorious, (c) exclusive, (d) hostile, and (e) each of the foregoing conditions must be continuous for the statutory period fixed by the state. Each of these terms needs some further definition.

Actual occupation means that written or oral statements claiming ownership are not valid. There must be actual and visible evidence of occupation on the ground.

Open and notorious occupation can be evidenced in different ways, such as a fence, cultivation of the ground, erection of buildings, posted boundary markers, etc.

Exclusive possession means that it cannot be shared with anyone. Thus, a driveway used in common with someone else could not be used as evidence for a claim under adverse possession.

Hostile possession does not necessarily imply ill will, but it means there must be no agreement between adjacent owners, or knowledge by the claimant of the true conditions. Accordingly, he will consider as a trespass any undue entry upon the property he claims.

Continuous occupation requires that there shall be no lapses or gaps during the statutory period of possession. Thus, occasional acts of ownership will not be sufficient to establish title, nor can two or more different persons at different times exercise the rights of ownership. A possible exception to the last requirement is that where the title to property passes through successive owners; i.e., if the property is sold to a new owner and he meets all of the other conditions of adverse possession, his period of occupany may be added to that of his predecessor. This procedure is called *tacking* and may extend through two or more adverse claimants to meet the statutory period necessary to acquire title. However, there must be no gaps between the successive periods of occupation.

In reviewing the conditions necessary to effect ownership under adverse possession, it is evident that this change of ownership is conditioned entirely on the bona fide intent and belief of the claimant. Thus, in the case of the two owners *A* and *B* mentioned above, it was supposed that each one believed the fence to be in its correct location. It frequently happens, however, that both owners of adjacent properties are not certain about the correct location of the boundary line. In such a case, if there is a mutual understanding between them that the boundary fence in place may not be the true line, but they agree to use it as the boundary, then as long as these conditions exist neither owner can claim ownership to the fence under adverse possession. But if at a later time *A* sells his property to another person *C*, who is not informed of the uncertain location of the fence and who believes it to be the true boundary, then *C* can claim ownership and title to the fence under adverse possession if he occupies his land according to all of the necessary conditions stated above. In the latter case the fence has become the true boundary line.

As between the public and individuals, it is a general principle that, although the government, either federal or state, may acquire title by adverse possession, the reverse is not true, i.e., the statute does not run against the state. Hence, individuals or private corporations cannot acquire public property by means of adverse possession.

12-35. Highways and Streets An engineer or surveyor frequently has occasion to deal with public improvements, such as pavements, sidewalks, and sewers, to be constructed in streets or highways, or he is required to establish property lines along a highway or street. He should, therefore, be fully informed of the conditions under which title is held for such property.

It is a common principle of law that title to property bordering on a highway or a street extends to the center of such highway or street unless there exists an explicit restriction. Such a restriction might be imposed by a metes and bounds description, or by the ordinance or statute under which the street or road was laid out.

Some statutes and city ordinances provide for no greater interest in a street by the public than an easement or right to its use as a thoroughfare, and title remains in the owners of the adjacent lots. Also, a street or a highway opened by condemnation proceedings may be governed by statutes which provide for an easement only by the public. Accordingly, when the street or highway is abandoned and the public can show no further need to use it as a thoroughfare, the right to use it reverts to the adjacent property owners.

However, some ordinances and statutes prescribe that title to property used as streets shall be held by the city, and similarly, that title to the right-of-way for improved highways shall be vested in the state.

12-36. Establishing Streets and Highways The manner of laying out of highways or city streets is always prescribed by state or municipal laws and ordinances. Because of the importance of the matter to all concerned, it is essential that the public authority, the owner, and the surveyor shall comply strictly with the legal requirements in establishing street or highway lines.

It should be noted also that a highway may become established through use only, and without having been formally laid out. If the public becomes accustomed to travel along a roadway until its use is deemed a necessity and if its use has been continued for a statutory period of time, the roadway becomes as permanently established as by any other means. This procedure has many phases similar to the principle of adverse possession, although the period of time is usually much less for the case of fixing a roadway than for establishing title by adverse possession.

12-37. Monuments It is important that highway and street lines shall be fixed by permanent markers, such as concrete posts, cut stones, or iron pipes. Such markers should be 3 or 4 ft long so as not to be affected by frost action, set flush or below the surface of the ground, and carefully referenced to any nearby permanent objects.

Where roadways are improved on section lines, the corner markers should be lowered 2 or 3 ft below the surface and referenced to other permanent objects, or by witnessing corners set inside the fence lines.

12-38. Liability A surveyor or a civil engineer belongs to a learned profession and, although that condition brings him certain privileges, it also imposes the responsibility of performing his duties with competence and honesty; if he is negligent in his work he may be held liable for damages suffered by his client. His responsibilities in this respect are similar to those of a doctor or a lawyer. Thus, if he is employed to make a survey for a building site and is told the character of the building to be erected, and his survey is carelessly done without regard to the circumstances, he may be held liable for damages to his client resulting from an erroneous location of the building.

He is not required to achieve absolute accuracy in his results, but he is obliged to exhibit that degree of care, prudence, and judgment which may properly be expected of a surveyor or a civil engineer under similar circumstances.

12-39. How to Make a Survey for a Deed When a parcel of land is to be sold for which a survey and a deed description are required, the procedure is somewhat as follows:

1. The descriptions of the tract to be surveyed and of adjacent tracts are obtained and carefully examined.
2. The corners of the tract to be sold are established by resurvey if necessary, as explained in previous articles.
3. A resurvey of the tract is made in which the following data are obtained: (a) the length of each side of the field, (b) the angles at each corner, (c) the kind and position of accessories at each corner, (d) the calculated bearing of each side, referred to true north if possible, (e) the names of adjacent owners.

4. From these data the area is calculated, a plat is drawn, and a new description is written.

The plat should show the following. (a) the length of each side, (b) the angle at each corner, (c) the calculated area of the tract, (d) the kind of markers set and the references to the accessories at each corner, (e) the names of adjacent property owners, (f) the positions of buildings, roads, sidewalks, or other permanent objects that would help to perpetuate the property lines, (g) a title and meridian, (h) a description of the property lines, (i) a surveyor's certificate.

An example of a plat is shown in Fig. 12-13.

Office Problems

12-1. Draw sketches to show the lines that would be shown on a government plat, and give typical dimensions, for sections 2, 6, 18, 21, and 31 of a given township.

12-2. The dimensions of the sides of section 6 for a given township are as follows: north, 76.47 chains; south, 76.27 chains; east, 80.44 chains; and west 80.00 chains. Find the dimensions and areas of each tract of this section as shown on a government plat.

12-3. The corner common to sections 21, 22, 27, and 28, is lost. The nearest adjacent corners in place are: A, 1 mile west; B, $1\frac{1}{2}$ miles east; C, $\frac{1}{2}$ mile north, and D, 1 mile south. The original plat dimensions are as follows: between sections 21 and 22 = 80.00 chains; between sections 27 and 28 = 80.00 chains; between sections 21 and 28 = 80.10 chains; between sections 22 and 27 = 80.14 chains; between 23 and 26 = 79.88 chains. The present measured dimensions are as follows: AB = 13,233.4 ft; CD = 7,911.6 ft. Find dimensions AO and CO to restore the lost corner.

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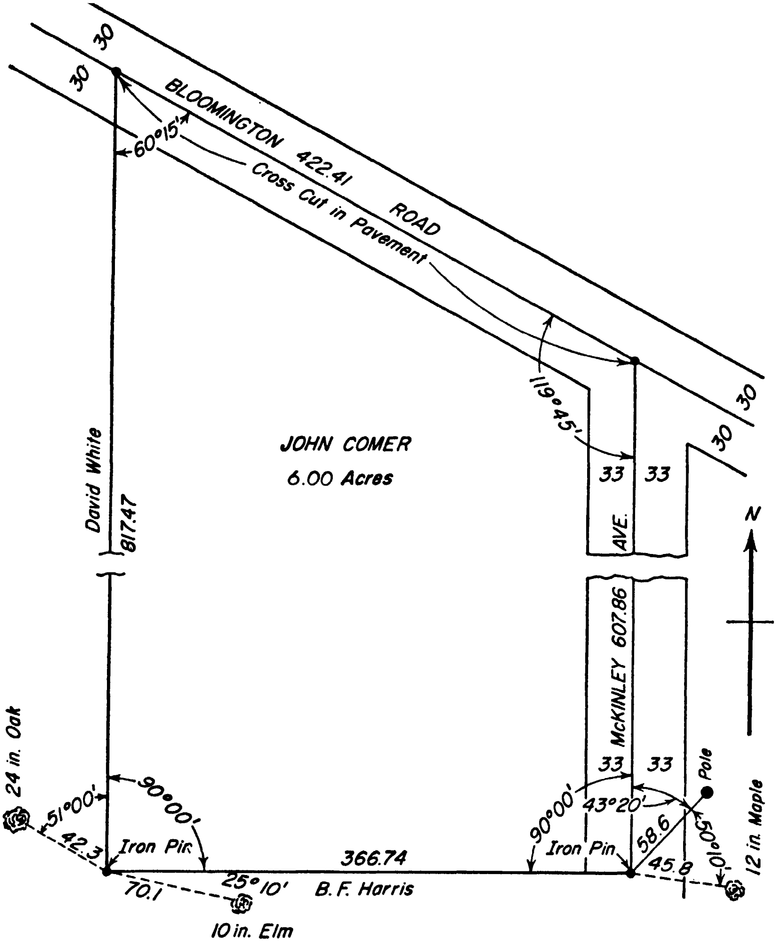


Fig. 12-13. Plat

County of Champaign
State of Illinois. } *s.s.*

I, Godfrey Sperling, County Surveyor of the County of Champaign and State of Illinois, do hereby certify that, at the request of John Comer, I have caused a survey to be made of the following described tract of land: commencing at a point 584.10 ft North of the S.E. corner of N.W. 1/4 of S.E. 1/4 of Sec. 2, T.19N., R.8E., 3rd P.M., thence North 607.86 ft to the centerline of Bloomington Road, thence in a Northwesterly direction along the centerline of said Bloomington Road 422.41 ft, thence South 817.48 ft; thence East 366.74 ft to the point of beginning; containing 6.00 acres, and marked the corners thereof as shown on the accompanying plat.

Dated this 6th day of May A.D. 19___ (Signed) Godfrey Sperling

County Surveyor

CHAPTER 13

TOPOGRAPHIC SURVEYING

13-1. Remarks The distinguishing characteristic of a topographic survey is that it results in a map which shows not only the horizontal dimensions but also the difference in elevation, or relief, of the earth's surface.

Various means have been used to show relief—color tints, hachures, shading, relief models, and contour lines, but the latter has such important advantages that it has become practically the universal method. The methods of constructing and using contour maps have been explained in Chapter 10. The purpose of this chapter will be to explain the proper methods of conducting the field surveys.

The scope of this subject includes the entire range of areas from the survey of a tract for a building site to the topographic map of the entire national domain. Also, the various purposes for which topographic surveys are made include the design of such projects as railway and highway location, power development, irrigation, drainage, water supply, municipal improvements, and the landscape developments of private estates and public parks. It is evident that it would be much beyond the scope of this chapter to attempt a comprehensive treatment of this subject. Accordingly, this discussion will be limited to surveys of relatively small areas for construction projects. This treatment, furthermore, is restricted to ground surveying methods, although the principles of photogrammetry discussed in Chapter 16 are widely used in the aerial photographic approach to topographic surveying.

The methods used in a given topographic survey are directly related to the scale to which the resulting map is drawn. Three classes of maps are commonly recognized—large-scale, intermediate-scale,

and small-scale maps. Large-scale maps include those drawn to scales between 1 in. = 10 ft and 1 in. = 100 ft; intermediate-scale maps are those drawn to scales between 1 in. = 100 ft and 1 in. = 1000 ft; small-scale maps include all others.

13-2. Horizontal Control by Traverse In topographic surveying it is necessary to establish over the area a network of instrument stations from which all necessary details may be observed. As stated previously, this system of horizontal control can be established either by traverse or by triangulation. If conditions are favorable to the traverse method, the instruments and procedure are as described in Chapter 6. The azimuth method is most commonly used, since this method greatly facilitates the location and plotting of details.

For small areas, the choice of location for control points will be simple and obvious. For more extended surveys it will be necessary to plan the route of the traverse so that the instrument stations will be properly distributed over the area, and that no traverse circuit will be too long.

If the area is several square miles in extent, it will be necessary to establish two orders of control, namely, third-order and fourth-order. Here it may be sufficient to say that third-order accuracy for traverse requires that the error of closure shall not exceed 1/5000, and fourth-order refers to any accuracy less than third-order. The third-order traverse should be run in circuits that close, either upon themselves forming closed loops, or upon other stations of higher-order control. The fourth-order control will consist of traverses of lesser accuracy run from one point on the third-order control to another, and establishing the instrument stations from which the details will be located.

The accuracy desirable for the fourth-order traverse depends upon the scale of the map and the use which it is to serve, and while no general rule can be given that will apply to all surveys, the following suggestions, based on the scale of the map, should be useful.

It is evident that the error in position of any plotted point on the map is the result of both field and plotting errors. Since the latter can be controlled more readily than the former, it is evident that care should be taken to reduce the plotting errors to a minimum. Then if field errors are rendered less than plotting errors, a satisfactory map will be obtained.

It may be assumed that 1/50 in. is a proper limit of error for any plotted control point on a map. If a total error of closure for a traverse is not more than double this amount, i.e., 1/25 in., then, when the survey is balanced, the adjusted positions of the instrument stations will, on the average, be much less than 1/50 in. Accordingly, if the total error of closure for a traverse circuit is not permitted to exceed an amount represented by 1/25 in., to the map scale, the accuracy of the control will be satisfactory.

For example, suppose the scale of a map is 1 in. = 100 ft and the control traverse is 1 mile long. Then 1/25 in. on the map corresponds to 4 ft on the ground, and the permissible error of closure is 4 ft, or 1/1300, which is only ordinary precision for a transit-tape traverse. If, however, the traverse were 5 miles long, then the permissible error of closure is the same, i.e., 4 ft, or 1/6500, which is relatively high precision. Again, if the scale of the map is 1 in. = 500 ft and the control traverse is 1 mile long, then 1/25 in. on the map corresponds to 20 ft on the ground and the permissible error is 20 ft, an adequate tolerance even for the stadia method.

13-3. Triangulation If the field conditions are favorable to a system of triangulation, the stations should be selected so as to form well-shaped triangles, properly distributed over the area. The methods and specifications for this work are more fully described in Chapter 6.

13-4. Vertical Control The vertical control consists of a system of benchmarks conveniently located for use when the elevations of contour points or other objects are to be determined. These are usually established on circuits run by the usual methods of differential leveling. Sometimes, if lesser accuracy is sufficient, stadia leveling may be used. The benchmarks are established at suitable intervals, usually less than one mile apart, and frequently the traverse stations are thus used.

The accuracy for most purposes will be satisfactory if the permissible error of closure is $C\sqrt{M}$, where C is 1/10 of the contour interval to be used, and M is the distance in miles. Thus, if the contour interval is 1 ft, the permissible error of closure in feet would be $0.1\sqrt{M}$. Also, when the contour interval is 10 ft, the permissible error is $1.0\sqrt{M}$, and this accuracy can be secured by stadia leveling.

13-5. Accuracy of Map The accuracy of a topographic map depends on four factors: (a) the number of points located, (b) the distribution of the points over the area, (c) the accuracy with which the observations are made, and (d) the skill of the topographer in generalizing the map from the given data. The first three factors will be discussed in the articles that follow, and the fourth refers to the ability of the topographer faithfully to represent the ground surface by the interpretation he gives to the contour lines. Little can be said about this important matter except that skill is to be gained largely by experience and that a very great advantage is provided if the contours can be drawn in the field while the terrain is in view. This is the most important consideration in favor of the plane-table method of mapping.

13-6. The Map Scale Many considerations may govern the scale of a map, but, in general, that will be a proper scale which is just sufficiently large to permit all desired features to be shown clearly and all dimensions to be scaled with the desired accuracy. Engineers, accustomed to the large-scale drawings of structures, are apt to choose a much larger scale for a map than is necessary. Possibly little harm results from such practice except that oftentimes dimensions are scaled from such maps with a precision which the accuracy of the map does not warrant.

13-7. The Contour Interval The proper choice of the contour interval for a topographic survey depends on the slopes of the terrain to be represented, the scale of the map, and the purpose of the survey. In hilly regions, if the contour interval is too small in relation to the scale of the map, the contour lines become so crowded as to become illegible and to obscure other features. Also, the accuracy and number of the field measurements should bear a consistent relation to the contour interval. Consequently, the cost of a map to a small interval is much greater, for a given area, than one to a large interval.

No definite rule can be given that will meet all conditions, but for most purposes and field conditions, the contour intervals for large-scale maps (1 in. = 10 ft to 100 ft) will be taken as $\frac{1}{2}$, 1, 2, or 5 ft for slopes which range from flat to hilly. For intermediate-scale maps

(1 in. = 100 ft to 1000 ft) and for corresponding slopes the contour interval may be taken as 1, 2, 5, or 10 ft.

13-8. Tests of Accuracy Two tests of accuracy are commonly applied to topographic maps, (a) test dimensions, and (b) test profiles.

The accuracy in position of points on a map may be tested by scaling from the map the distances between selected points and comparing these with the corresponding distances measured in the field. Maximum and average discrepancies are usually specified, the limits being fixed by the purpose of the survey.

The accuracy of the contour lines on a topographic map is tested by the comparison of two profiles, one of which is scaled from the selected route on the map, and the other plotted from the profile levels run over the same route on the ground. Maximum and average discrepancies between points at regular intervals along the profiles are specified.

13-9. Accuracy Required It is generally agreed that the elevation of a plotted contour point is sufficiently accurate if its value is correct within one fifth of a contour interval. The sources of error which affect the determination of the elevation of such a point are (1) measuring the azimuth, (2) the distance, and (3) the vertical angle or the direct rod reading. In any survey, therefore, these sources of error should be considered to the end that satisfactory results shall be obtained, and that time shall not be wasted on unnecessarily refined measurements.

Here, it should be remembered that 01′ of arc is 0.0003 radian (nearly) and that this value applies to vertical as well as to horizontal angles. Hence a 01′ error in the azimuth of a point will displace it 0.3 ft at a distance of 1000 ft; likewise, an error of 01′ in the vertical angle to a point 1000 ft distant will introduce an error of 0.3 ft in its elevation. It is evident, therefore, that precision in the measurement of the vertical angle is of much greater importance than the measurement of the azimuth. In fact, it may be stated as a general rule that, in measuring the azimuths of contour points, the angles need not be read more accurately than to the nearest 05′.

Reference to Art. 9-10 will show that, for gentle slopes, the error in the elevation of a point determined with an inclined sight, due to

the measurement of the vertical angle, will be more serious than that due to reading the stadia distance; but that, for steep slopes, the measurement of distance becomes of greater importance than the vertical angle.

The above remarks may be summarized by listing the three sources of error in the order of their importance from the greatest to the least: (1) measuring the vertical angle, (2) reading the stadia distance, and (3) measuring the azimuth.

13-10. Specifications for a Topographic Survey The principles considered in the preceding articles of this chapter may now be used to prepare specifications for each part of a topographic survey. For example, it may be assumed that specifications are desired for the field work for a topographic survey of a city park for which the conditions are as follows: the area is one square mile, the scale is 1 in. = 100 ft, the contour interval is 2 ft, and the ground slopes are about 5%.

Horizontal Control.—It may be assumed that the horizontal control will be a transit-tape traverse established within the boundary and will be about 3 miles long. The permissible error of closure will be 1/25 in. × 100 ft = 4 ft. When the error of closure is computed, adjusted, and plotted, all control points should be in their correct position within 1/50 in.

Vertical Control.—It is assumed that the vertical control will be a line of levels about 3 miles long, and, since the contour interval is 2 ft, the permissible error of closure will be $0.2 \text{ ft} \sqrt{3} = \pm 0.34$ ft.

Contour Points.—The requirement for a contour point is that its position on the map shall represent an error in elevation not greater than one fifth of the contour interval. Then if the ground slope is 5% and the contour interval is 2 ft, the map distance between contours will be 40 ft = 0.4 in. and one fifth of the distance is 8 ft = 0.08 in.

The position of a contour point must be fixed in three dimensions —*distance, azimuth,* and *elevation.* The point of reference is the transit station and the line of reference is the reference meridian.

Since the ground slope may have any direction, it will be necessary to specify an accuracy for each dimension measured, i.e., distance, azimuth, and elevation, such that the resulting error in the measurement will not exceed one fifth of the contour interval.

Distance.—According to the above conditions the distance to a contour point should be correct within 8 ft, and since the distances

are nearly always stadia measurements, this means that the stadia intercept should be read correctly within ±0.08 ft.

Azimuth.—The error in the azimuth measurement will cause a displacement perpendicular to the line of sight where the contour point is observed, and this displacement should not exceed 8 ft on the ground. Since azimuth is measured in degrees of arc and since the tangent of $01' = 0.0003$ (very nearly) the permissible error in minutes of arc e_a, for a given distance D, will be given by the relation $e_a = 8/D \times 0.0003$. So if the distance to a contour point is assumed to be 900 ft, the permissible error in the azimuth reading would be $e_a = 8/900 \times 0.0003 = \pm 30'$ (nearly). This is seen to be a large permissible error on the vernier of a transit and indicates that time need not be wasted in reading the vernier with unnecessary precision.

Elevation.—The permissible error in elevation for this example, as stated above, is 0.4 ft. Accordingly, when a vertical angle is read at a contour point, the permissible error in minutes of arc is $e_a = 0.4/900 \times 0.0003 = \pm 1.3'$. This indicates that the vertical angle must be read with much care.

Of course, the values of the permissible errors computed here for distance, azimuth, and elevation will change proportionally with the distances to other contour points.

From the preceding analysis and computations the field party now have definite specifications for each step in the survey which should yield an adequate map with a minimum of time and effort in the field work.

13-11. Selecting Contour Points The field work of finding the contour points for each of the four systems (Art. 10-3), except that for system C, is rather mechanical and offers little opportunity for the exercise of judgment on the part of the rodman. Where controlling points are being selected, however, the rodman is required continually to use his best judgment in selecting the points to be observed. In general, these will be at the marked changes in slope and along ridge and valley lines. However, small irregularities in the ground surface cause the rodman to consider which points shall be observed and which shall be disregarded.

A general principle, which should be of aid, may be stated. Obviously the magnitude of irregularities which may properly be disregarded will depend upon the accuracy specified for the completed

map. If, then, it is assumed that the average error of the test profile points shall be not greater than one half of a contour interval, it will be necessary to select contour points such that an imaginary straight line joining any two adjacent points will not pass above or below the ground surface more than one contour interval.

For example, let it be supposed that a ground profile is shown as *a*,*b*,*c*, Fig. 13-1, where two contour points have been located at points

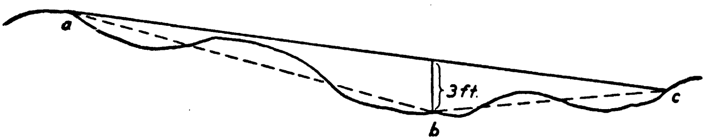


FIG. 13-1. Contour Points.

a and *c*, perhaps 200 ft apart. The rodman must judge whether one or more intermediate readings should be taken. If no intermediate reading is taken, the map contour lines would show the uniform slope *a*-*c*, which passes above the ground 3 ft at point *b*. If, then, the contour interval is 2 ft, evidently a reading should be taken at point *b*, after which the map contour lines would show the dotted slopes *a*-*b* and *b*-*c*. The deviations from these lines are each less than a contour interval, and hence the other irregularities along the profile can be disregarded.

13-12. Location of Details The location of details is conditioned by four factors, (a) the scale of map, (b) the instrument used, (c) the system of contour points, and (d) the character of the terrain. Obviously these factors vary so widely that it would be impossible to discuss all the varieties of field conditions which occur. Accordingly, the subject will be treated with respect to each of the systems of contour points described in Art. 10-3.

13-13. System *A*—Coordinates The system of points shown in Fig 10-3 (*a*-1) and (*a*-2) is used where an intermediate degree of accuracy is desired and is the one most commonly used for large-scale maps. The methods of establishing the points vary with the field conditions, three of which are illustrated in Fig. 13-2.

Method 1 consists in establishing coordinate axes as *o*,3-*f*,3 and *c*,*o*-*c*,6 with their origin at *c*,3. Along these axes, stations are set at regular intervals, usually 100 ft, as *d*,3, *e*,3 and *f*,3. From these sta-

tions, lines are run to establish a grid system as shown in the upper left-hand quadrant of Fig. 13-2.

Method 2. By this method, a rectangular traverse is run about the perimeter, setting stakes at, say, 100-ft intervals. The interior stakes may then be set by sighting from a point a,o on one side to the corresponding point $a,6$ on the other side, and measuring the intervals by stadia or tape.

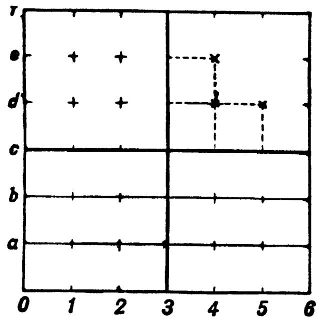


FIG. 13-2. Methods of Establishing Coordinates.

Method 3 is advantageous in regions of dense vegetation where sighting with an instrument is difficult. Three tapemen with two tapes proceed as follows: coordinate axes are first established, then beginning at the origin as $c,3$ in the figure, one tape is stretched from $d,3$ and the other from $c,4$ and the ends brought together to locate point $d,4$. In similar manner, the other interior points are located, such as $e,4$, $d,5$, etc.

The positions of all points, located by either of the methods described above, are readily checked by the condition that the distances between adjacent points should be the same throughout the system.

The elevations of the coordinate points may be found by either the engineer's level or the hand level, depending on the accuracy desired. If the area is extensive, it is best to find the elevations along the axes or about the perimeter with the engineer's level, then the hand level may be used for the interior points, since checks are provided at frequent intervals thus preventing large cumulative errors. The precision of hand-level work has been stated in Art. 3-13.

On irregular terrain the accuracy of the map will be greatly increased if the contours are drawn in the field either on a sketch board or a plane table.

13-14. System *B*—Tracing Out Contours The most accurate method of locating contour lines is that of tracing out the lines upon the ground. This is done most efficiently by the combined use of the transit and level. Of course, a plane table and level may be used, but since the contours are located by finding points on the lines them-

selves, the principal advantage of the plane table is much reduced.

Both the horizontal and vertical control are established by a traverse and line of levels as may be required. A transit is set up and oriented; also a level is set up where it will be most convenient in relation to the contours to be located. The levelman directs the rodman until he finds a point on the desired contour. The transitman then locates the position by means of an azimuth and a distance reading. The rodman then proceeds along the same contour until a marked change in direction occurs, where a point is found and located as described above. In this manner the contour is traced out on the ground as far as the positions of the instruments will permit. The levelman then directs the rodman to a point on the next adjacent contour and the process is repeated.

A hand level is convenient for this work, but of course the accuracy would not be as great as where the engineer's level is used.

For both Systems *A* and *B* it is desirable that permanent stakes and benchmarks be established during the survey to serve as reference marks during the subsequent construction work.

13-15. System *C*—Controlling Points. The Transit Method This system of points consists of such controlling points as summits, ridge and valley lines, and all important changes in slopes. The points are located in the field by the use of either the transit or the plane-table instruments. The procedure with each instrument will now be described.

The relative merits of the transit and plane-table instruments have been discussed in the previous chapter. It may be added that, because of lack of experience, many engineers fail to appreciate the important advantages which the plane table possesses. However, since most engineer offices are not equipped with one, the transit has a wider use in topographic surveys for construction purposes than does the plane table.

The transit party usually consists of three men—an instrumentman, a recorder, and a rodman. Sometimes two rodmen and one or more axmen are employed. The organization of the party should be flexible to meet the field conditions. Thus, the three-man party may be organized as a transitman, recorder, and rodman; or, if the distances are great, it may be organized as a transitman and two rodmen.

For the usual field conditions, the procedure of locating details is

somewhat as follows: the instrumentman sets up over a control station and orients the transit by a sight on an adjacent station, and determines his H.I. above the ground. The rodman selects a point to be located and presents his stadia board for a reading. The transitman sights the board using the upper motion of the transit; sets the bottom cross-wire on a full foot mark, and reads the stadia interval by noting where the upper cross-wire cuts the rod. He then sets the middle cross-wire at a point on the rod which has the same height above the ground as the H.I. of the instrument, and motions the rodman to another point. While the rodman is finding another point, the instrumentman reads the vertical angle and the azimuth of the point observed. The recorder keeps the notes and, if time permits, computes the elevations and horizontal distances of all inclined sights. Sketches are often better than written descriptions of points and should be used freely. This procedure is varied somewhat if direct level readings are possible.

The rodman proceeds in a systematic route over the area to prevent the omission of important details. He also takes notes of significant features, which supplement those that the instrumentman is able to observe.

13-16. The Plane-Table Method The organization of a plane-table party ordinarily consists of an instrumentman, a computer, and a rodman, but unusual field conditions may require some modification of this organization. Since the map is constructed in the field, no permanent record of observations is necessary, but the proper reductions of the readings for plotting are required. As stated above, the controlling-point system of points is used, and the general use of the plane table has been described in Chapter 11.

Skill in sketching contours and drawing the map in the field will be gained with experience, but one suggestion may be made here. Objects near the instrument are more plainly visible than those at a distance, and the beginner is likely to spend an inordinate amount of time in representing with great precision the features near his instrument, while overlooking or improperly sketching more significant features at a distance.

The time spent in the field by the plane-table party can be economized if the topographer will omit whatever finishing touches have no effect on the completeness or excellence of the map. Thus, when the controlling points of a feature have been located, the contours

properly spaced along the ridge and valley lines, and perhaps every fifth contour is properly drawn, the remaining work of drawing the intermediate contour lines may be omitted in the field and completed in the office. Care will need to be exercised in this matter, however, to be sure that all essential lines are drawn in the field.

13-17. System *D*—The Hand-Level Method The method used for locating points according to System *D* makes use of the hand-level instrument. The field party usually includes the topographer and two tapemen. The equipment consists of a hand level, a 100-ft steel tape, a rod (10 to 15 ft long) graduated in one-foot divisions, and a 5-ft staff. The record is kept in the regular transit notebook, or in a special topography book made up of cross-ruled pages.

This method is used principally in connection with route surveys and hence it is assumed that a located line has been established by means of a transit traverse, stakes being set at 100-ft intervals, and that profile levels have been run over the line.

In the field, cross sections are taken at the 100-ft stations or at irregular intervals, depending on the conditions. The procedure may be described by reference to Fig. 13-3. At station 40 the elevation

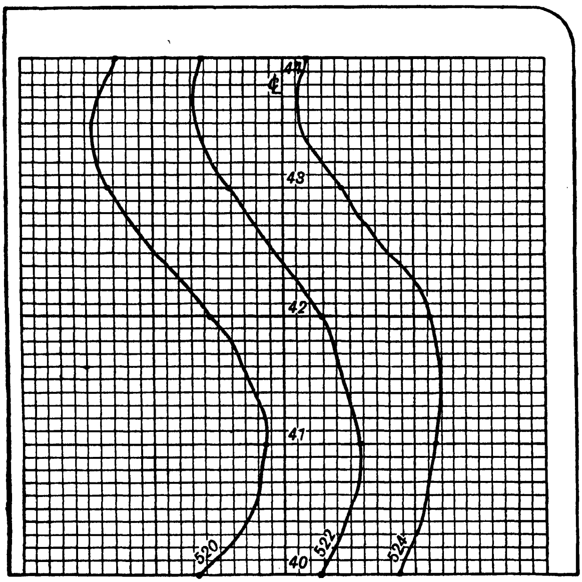


FIG. 13-3. Locating Contours by Cross Sections.

(521.5) of the ground at the stake is known from the profile levels. Assuming that the contour points to the right of the centerlines are to be located, the head tapeman takes the hand level and moves out on a line perpendicular to the centerline. The rear tapeman takes the rod and remains at the center stake. The ground slopes uphill and, by sighting on the rod held at the center stake, the head tapeman is directed to a point which is 0.5 ft higher than station 40, and thus locates a point on the 522-contour. The distance from the center stake is 27 ft, and this point is plotted by the topographer in his book. The rear tapeman then takes the position at this point and the head tapeman goes forward to find a point on the next higher (524-ft) contour. The distance between the two contours is 58 ft, or the latter point is 85 ft from the center stake. The method of taping may be that of measuring from one contour point to the next, or the topographer may serve as rear tapeman, thus permitting the rear tapeman to become a rodman, and all distances are measured continuously from the center stake.

In locating points to the left of the line, the slope is downhill, and either the tapemen reverse positions, or the head tapeman carries the rod and the rear tapeman the hand level.

Distinguishing marks are usually placed on the rod at each contour interval above and below the H.I. of the hand level. The H.I. of the hand level may be the natural height of the levelman's eye above the ground, or a 5-ft staff may be used to support the hand level. The latter practice is better, because it permits the tapemen to reverse positions without confusion.

13-18. Applications Some applications of the methods described above may now be indicated. Systems *A* and *B* are used where a relatively high degree of accuracy is desired, and hence the map is usually drawn to a large scale. Such surveys are required for the location and design of important structures, such as dams, reservoirs, bridges and buildings; and for the earthwork estimates for such projects as roadways, levees, canals, and landscape grading. System *B* is used for the highest accuracy, or when the slopes are gentle. On rolling or hilly terrain the number of points to be located for this system is relatively high, and the cost is, therefore, correspondingly high. However, if high accuracy is desired, this method is used, regardless of the character of the terrain.

System *C* undoubtedly has the widest application of any of those

here described. It is suitable for the many surveys for intermediate-scale maps which serve for the design and estimates of many engineering projects, such as municipal improvements, reservoirs, irrigation, drainage, and hydroelectric developments.

System *D* is especially suitable for the route surveys for such projects as highways, railways, drainage ditches, and canals. It is also well adapted to any survey that encounters a terrain covered by dense vegetation, because the hand-level instrument is especially advantageous under such conditions.

13-19. The Building-Site Survey When the design of a building is undertaken, the architect needs certain information regarding the situation. This is sometimes provided by field notes, but is better shown on a plat. Such a plat is best obtained on a plane table in the field. The data needed can be classified in two groups: (1) elevations and (2) objects.

Elevations.—The elevations of the following points are needed: (1) at intervals of 50 ft along the pavement curb, also at 50-ft intervals along the inside edge of the sidewalk; (2) contour points at the corners of 50-ft squares over the area; (3) inverts of the sewer outlets from manholes and the calculated gradients of the sewers; and (4) the reference benchmark with a description of the same.

The elevations of all ground points are read to tenths of a foot, whereas all others should be read to hundredths. Contour lines may be drawn if the ground is quite irregular.

Objects.—The objects to be located include the following: (1) the lot corners (state materials used); (2) the property lines, give dimensions; (3) the street lines, give widths; (4) pavements, give widths back to back of curb, and materials used; (5) sidewalks and drives, state kind and widths; (6) exact location of gas and water mains, give size; (7) manholes; (8) poles; (9) trees, state kind and size; (10) water hydrants; (11) storm and sanitary sewers, give size and kind; and (12) existing structures, give size, kind, and location.

The map should also show a legal description of the tract and a description of the reference benchmark to which all elevations must be referred. The drawing is made on tracing cloth so that prints may be made to accompany the architect's plans.

CHAPTER **14**

MAP DRAFTING

14-1. Remarks The distinction should be made between a plat and a map. A plat is a dimension drawing showing the data which pertain to a land survey or a subdivision of land. A map is a graphical representation, to scale, of the relative positions and character of the features of a given area. A plat is primarily a legal instrument being made a matter of public record and used in the description and conveyance of real estate, having all important dimensions recorded. Although a plat is usually drawn to scale, little if any use is made of this condition. A map, however, shows no recorded dimensions, but a large part of its usefulness depends on the accuracy with which distances, areas, and elevations can be derived from it.

Much that is said in this chapter applies to maps and plats alike, but the principal considerations will be the construction of maps for engineering purposes.

From what has been said above, it is evident that the accuracy with which the relations between points are shown on a map is a prime consideration. This accuracy depends both upon the field work and the drawing of the map, and since it is much more difficult to increase the accuracy of the field work than that of plotting the map, it is evident that the work of drawing the map should be careful and accurate. This accuracy is of a distinctly higher order than that used on mechanical drawings.

Excessive ornateness is not desired on maps that are to serve practical purposes; nevertheless the draftsman should strive for such harmonious effects that the finished map will have a pleasing appearance. This is attained by the use of good taste in the matters of form, proportion, and color in the symbols and lettering used.

14-2. Lettering One of the first impressions of the young engineer's work is that made by the appearance of his field notes or his

drawings. It is important, therefore, that he should devote a reasonable amount of time to make his lettering as excellent as possible. Natural ability is an important factor in this matter, but with sufficient patience and practice, any student can develop a reasonable proficiency in his lettering.

It is assumed that detailed instructions have been given elsewhere, but a few general suggestions are given below:

1. Guide lines should be used for all lettering on maps. Proficient draftsmen sometimes omit the upper two lines on subordinate parts of titles or in remarks, but at least a bottom guide line is always used.

2. The letters should be well formed, following closely the accepted patterns illustrated in Figs. 14-1 and 14-3.

3. A uniform slant is important, because even if the separate letters are well made, the general appearance of a body of notes will be bad if the slant is not uniform throughout. The slope ratio for slant letters is about $2\frac{1}{2}$ vertical to 1 horizontal. Vertical letters should have a slight inclination to the left or be truly vertical.

4. Either slant or vertical letters may be used, but both kinds should not be used in the same title or body of notes. Most persons make slant letters with greater facility than vertical; consequently, slant letters are quite universally used for field notes and for remarks and dimensions on drawings, while vertical letters are used for titles.

5. The size of the letters in titles should be proportional to the drawing as a whole. The ratio of lower-case letters to capitals is about 3 to 5, and in field notes the height of capitals is made equal to about $\frac{1}{2}$ the space between the ruled lines.

6. The spacing between letters should be narrow, a common fault being to make this spacing too wide. Also, the area between letters should be uniform. If a wide space is to be filled with lettering, the extended form may be used, and the space between the words may be increased, but that between the letters should remain narrow.

Mechanical lettering is now used very widely for map drafting. This practice has important advantages when more than one person performs drafting tasks in the preparation of the same map.

14-3. Styles of Lettering Three styles of letters are commonly used for notes and maps: *Reinhardt, Gothic,* and *Roman.* These are illustrated in Fig. 14-1.

Reinhardt, "single stroke" letters are most easily and rapidly

FIG. 14-1. Styles of Letters for Maps.

made and are standard practice for field notes and notations on maps and drawings. The weight of the letters, i.e., the width of the lines forming them, may be varied by the use of the different pens listed below.

Gothic letters are used in titles where a heavy weight and finished appearance are desired. These letters are executed with relatively fine pen by sketching carefully the outlines of each letter and then by filling in the middle space. By this procedure a more precise letter is formed than by using a single stroke and a coarse pen. These letters are often made with single strokes, using a suitable pen, but careful execution is required to secure good results.

Roman letters are used when diversity is desirable and where elegance and beauty are important considerations. These letters are difficult to make and some degree of expertness is necessary if good results are to be attained. They are distinctive, however, and their good appearance, when well made, gives them a widely prevalent usage.

Relatively little freehand lettering is employed today in the preparation of maps. Most of the names, symbols, marginal data, etc. which are to go on the map are printed on a tough, transparent, acetate sheet. It is then a simple matter to cut out the required name or symbol from this sheet, put it in the proper position on the manuscript map, and burnish it down. Transparent, pressure-sensitive adhesives on the back of the "stick-up" eliminate the necessity for applying a cement (see Fig. 14-2).

14-4. Titles and Meridians A title for a map usually provides the following items of information: (1) the organization or company for whom the map is made, (2) the name of the tract or feature which has been mapped, (3) the name of the engineer in charge, (4) the name of the draftsman, (5) the place or office where the map was drawn, (6) the date, and (7) the scale, both numerical and graphical.

In executing a title, proper emphasis should be given to the different items listed above by varying the weight and the size of the letters used. A principal purpose which a title serves is to identify a given map in a file with other similar maps. Hence, the most important item in the title is the name of the tract or feature which the map represents, and this item should be given the most emphasis. For example, in Fig. 14-3d the feature shown on this map is that

portion of the location between stations 425 and 563 on Route 47. Accordingly, this item is given the most prominence in the title.

A map title is placed anywhere on the drawing where it will balance the map as a whole. It is not boxed-in at the lower right-hand

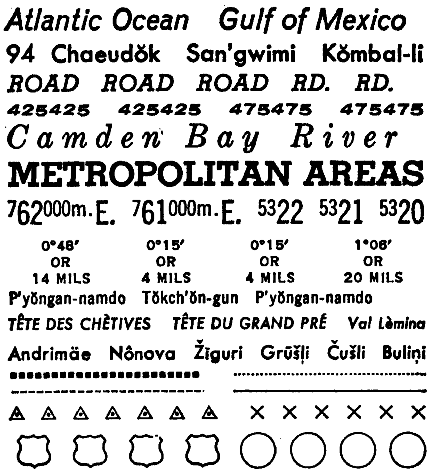


FIG. 14-2. Adhesive Map Type.

corner as are titles on mechanical drawings. The size of the letters should be consistent with the size of the drawing. A general tendency with beginners is to make the title too bold, with letters too large or the weight of the lines too heavy. Another common fault is too much space between the lines, which gives the title a loose and disjointed appearance.

What has been said about lettering has special importance with regard to titles. Either vertical or slant letters may be used, but both kinds should not be mixed in the same title. Each title should be evenly balanced about a vertical axis through it. This often requires the title to be lightly sketched in pencil first, and properly adjusted before inking. The principles stated above are illustrated in Fig. 14-3.

A meridian should appear on every map. It is indicated by an arrow, somewhat as shown in Fig. 14-4b, which, unless otherwise specified, indicates true north. On plats, it is sometimes desirable to show both the true and the magnetic meridians and the angle of

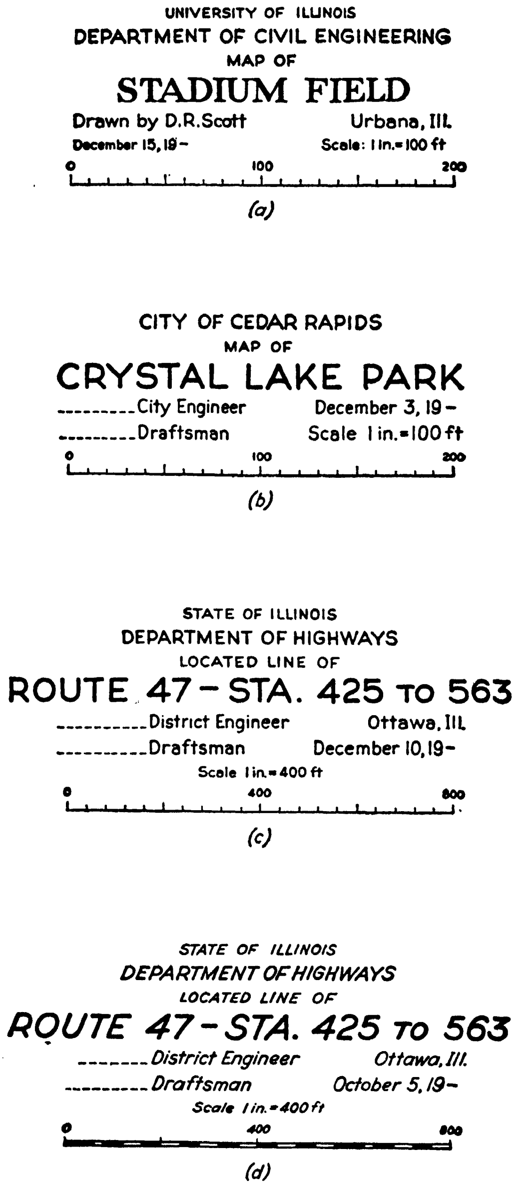


FIG. 14-3. Map Lettering.

declination, as shown in Fig. 14-4a. The south half of the arrow should be somewhat longer than the north half to give it a balanced appearance. A common fault is to make the meridian too bold and conspicuous by making it too large and heavy.

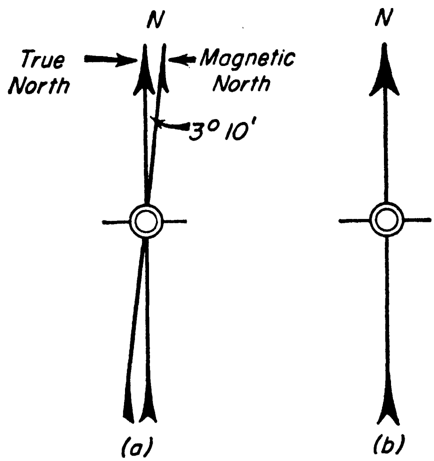


FIG. 14-4. Map Meridians.

14-5. Map and Drawing Scales Map scales are stated as the ratio of a given distance on the map to the corresponding distance on the ground. Thus a given scale may be expressed as 1/1200, i.e., 1 in. to 1200 in., or the same scale may be expressed as 1 in. = 100 ft. The map distance is stated first.

A large-scale map is one in which the ratio mentioned above is large compared with one for which the ratio is small. For example, 1 in. = 20 ft is a large scale, and 1 in. = 1000 ft is a small scale.

Architects use the relation between fractional inches on the drawing to 1 ft in the structure; as $\frac{1}{4}$ in. = 1 ft. Accordingly, a drawing scale which is divided into inches, quarters, eighths, etc., is called an architect's scale. The engineer's map is usually drawn to a scale in which 1 in. represents a decimal number of feet, and, hence, a scale which is divided into decimal parts of an inch, as tenths, twentieths, thirtieths, etc., is called an engineer's scale. An architect's scale is not suitable for drawing a map.

14-6. Symbols Symbols are used to portray various features on maps. Different draftsmen may use different symbols to represent

the same feature; it is desirable, therefore, that the most common features be represented by symbols which are widely accepted and understood. In this matter it is natural to follow the practice of those governmental organizations which are engaged in mapping work, such as the U.S. Geological Survey, the U.S. Coast and Geodetic

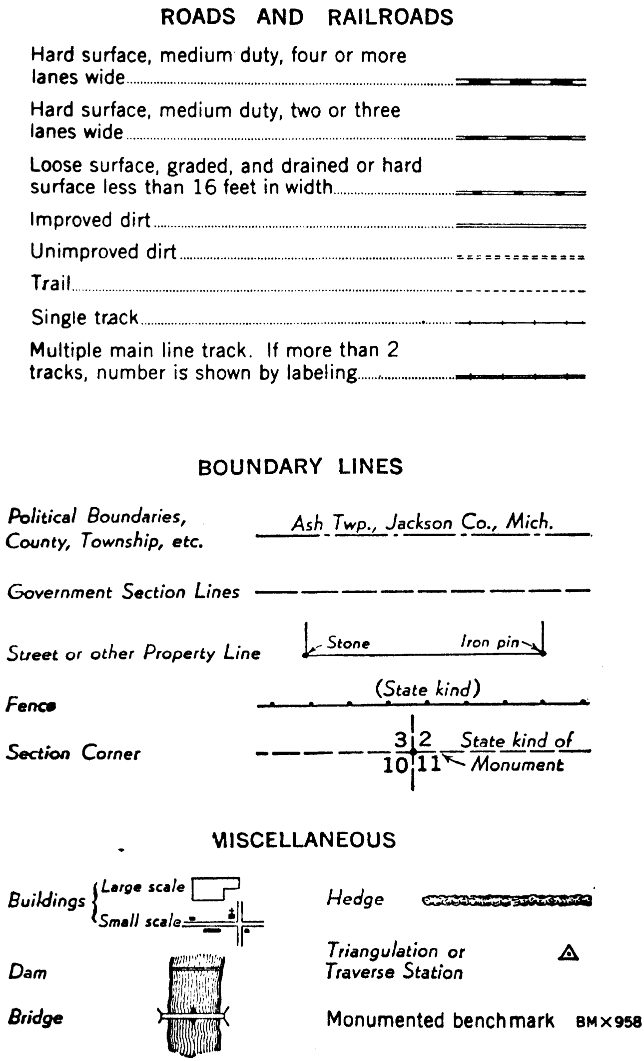


FIG. 14-5a. Map Symbols.

Survey, and others. The symbols shown in Fig. 14-5, a and b, are ones most commonly used.

Since map scales vary so widely, the character of the symbols will vary somewhat to suit the scale of the map. Thus, on large-scale maps, tree symbols (in plan) may be drawn to scale, while on small-scale maps no attempt is made to draw them to scale. Care must be taken in executing the symbols that they shall not obscure and render illegible other features on the map.

TOPOGRAPHY

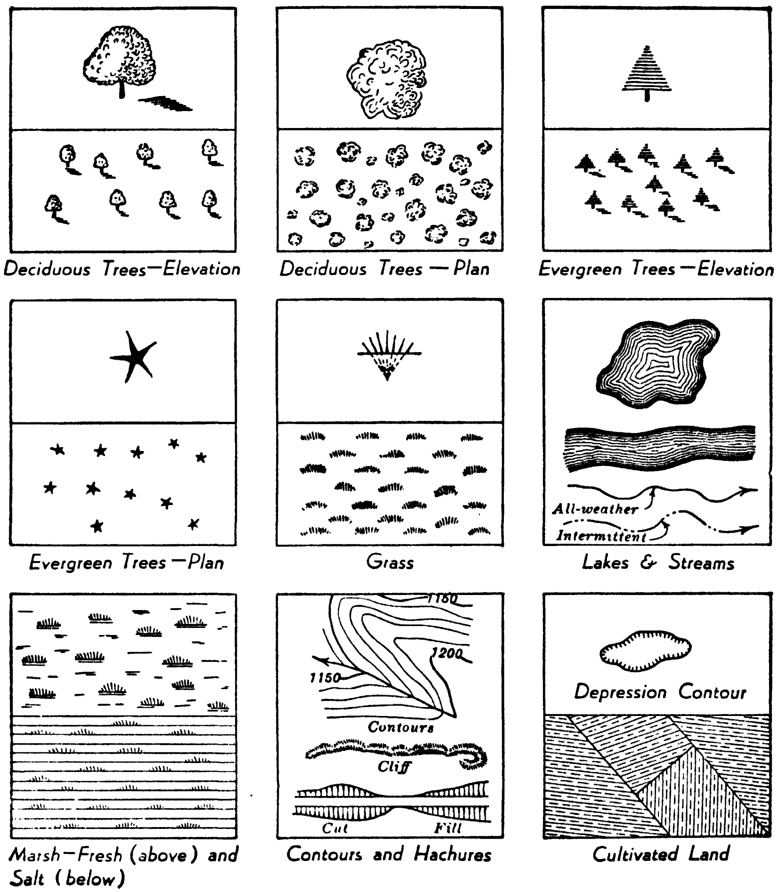


FIG. 14-5b. Map Symbols (cont.).

The matter of colors and the symbols to which they apply is discussed in the following article, but the manner of executing certain symbols needs detailed explanation:

Trees.—Distinction is commonly made between deciduous and evergreen trees, and each may be drawn either in elevation or in plan. On landscape maps the kind of tree and diameter of trunk are sometimes recorded on, or near, the symbol. It is done in black or green color.

Grass.—This symbol represents the blades of grass emanating from a root below the surface of the ground. The blades are drawn as straight lines whose tops form an even arc and whose bases are cut off on a straight invisible ground line. The number of lines varies, and the separate symbols should be irregularly scattered to prevent the appearance of rows. The size of the blades and the closeness of spacing of the symbols will depend on the scale of the map and the area to be covered.

Water Lines.—For lakes and wide streams the shore line is drawn slightly heavier than the inner lines. Next to it a fine line is drawn as close as may be, following all of the irregularities of the shore line. The next inner line is spaced slightly greater, and at small indentations in the shore line, the water line is given a sharp angular break in direction. The spacing between lines increases gradually away from the shore, and the breaks are made to fall on a straight line perpendicular to the shore line, thus giving the impression of wave action. It should be noted that the points of the water lines are always toward the shore.

In a wide stream where there are no small irregularities in the shore line, the water lines are not given sharp breaks but follow closely the shore lines. Toward the center of streams and lakes, the water lines may be broken to give the effect of open water. On very wide water surfaces the water lines are omitted in the central area. The good appearance of water lines depends largely on the evenness of the lines and the quality of spacing between the lines.

Marsh.—Fresh marsh is made by first drawing the grass symbol, then drawing a double water-surface line just below it. The intervening space can be filled in with small, broken, horizontal water lines. Salt marsh is drawn by spacing irregularly the grass symbol on ruled, closely spaced, horizontal lines.

Contour Lines.—The construction and drawing of contour lines have been discussed in Art. 10-4. A principal difficulty in drawing

these lines is to get a uniform weight, and for this purpose a specially devised "contour pen" is sometimes used. However, with a little practice and experimentation with pens of varying fineness, good lines may be secured freehand. The fifth lines are made heavy to facilitate reading the map.

Sufficient lines should be numbered to enable anyone to read elevations conveniently. The lines are broken and the numbers are written in the interval as shown.

A closed contour line on a map usually indicates a summit. Sometimes, however, a depression is present without a pond of water or other means of telling that it is a depression. In this case a special symbol is used as shown.

A common fault in drawing contour lines is to make them too heavy.

Hachure Lines.—Hachure lines are short even lines drawn parallel with the direction of steepest slope and with width and length varying proportionately with the degree of slope. Short heavy lines indicate steep slopes, and long thin lines indicate gentle slopes.

Hachures are little used on engineering maps now except to represent cliffs or earthwork cuts and fills for roadways, levees, etc. The hachure lines are modified somewhat for earthwork slopes, for which the line is made to extend the full length of the side slope and is widened slightly at the top of the slope, thus making it possible to identify a cut or a fill.

Various commercially produced plastic sheets like Zip-A-Tone, on which is printed drainage and woodland patterns, are extensively used. Such techniques make possible uniformity of appearance of the finished map and saving of time and money through the elimination of the need for more expensive hand lettering skills.

14-7. Ink, Water Color, and Pens On topographic maps it is quite necessary to draw the various symbols in different colors. The standard practice is as follows: black for lettering and the works of man, such as roads, railways, houses and other structures; burnt sienna (reddish brown) for all land forms, i.e., contours and hachures; prussian blue for water features, such as streams, lakes, marsh, and ponds; and green for vegetation, including trees and grass.

Either ink or water color may be used, but the variety of colors in drawing ink is limited and does not permit the use of the proper hues

on topographic maps. Water color, however, may be used to produce any hue or tint desired, but it is not as easily handled as ink, and some expertness and training are necessary to secure the best results. Accordingly, it may be said that ink will generally be satisfactory for engineering drawings, while water color is more suitable for large topographic maps or those used by landscape architects. It should be added that if colors are desired on tracings, the water color is more suitable because it is opaque, whereas the ink is somewhat transparent.

It is impossible to specify the degree of fineness or coarseness of lettering pens to be used for any purpose because of the difference in touch of the persons using them, but the following list, in order from fine to coarse, will meet most mapping requirements.

Lettering and Mapping Pens:

Crow quill, and Gillott's 290 for hairlines.
Gillott's 303, 404, for fine and ordinary lines.
Hunt 512 ball point, and Leonardt 516 ball point
for large lettering and titles.

14-8. Plotting. Remarks Two methods are used in plotting points on maps. By one method, a point is plotted by scaling a distance and laying off an angle; by the other, a point is plotted by scaling its rectangular coordinates with respect to some origin. In the following paragraphs the two methods will be described and their relative advantages compared. It is assumed that all important errors and mistakes in the field work have been removed before plotting is begun.

Many degrees of precision are used in plotting, depending on the purpose which the plotted point serves, i.e., whether it is a point on the horizontal control or a detail of the map. For most engineering maps, sufficient accuracy will be attained if points are plotted with maximum errors in position as follows: (1) control points, 1/50 in.; (2) definite detail points, such as corners of buildings, 1/30 in.; and (3) rough details like shore lines, 1/20 in. or less.

It is evident that if the maximum error in the plotted position of a control point is to be 1/50 in., the draftsman must exercise great care in the manipulation of his drawing instruments. In particular, he must have a hard (4H at least) pencil, sharply pointed. About $\frac{1}{3}$ in. of the lead should be exposed and pointed with a file or sandpaper. The triangles and straight edge should be carefully tested to

see that the edges are true and the angles exact. For map drafting, the T-square has little use except as a straight edge. When used on the ordinary drawing board or drafting table, it does not afford sufficient accuracy, as a T-square, for plotting control points.

14-9. Plotting Angles An angle is usually plotted by use of (1) a protractor or (2) the tangent function of the angle.

The Protractor Method.—A protractor is a circular disc or ring having its limb divided into 1° divisions or smaller. Protractors are made of various materials, but for mapping, that kind in which the circle of divisions is printed on a sheet of cardboard or heavy drawing paper is convenient.

The precision with which an angle may be plotted depends on the size of the protractor and the skill of the draftsman in estimating and marking subdivisions. If it is assumed that a point can be scaled within 0.01 in., then the angular error of plotting a point with a protractor of given size can be computed. For example, a protractor of 6-in. diameter has a circumference of 18 in. (nearly). Hence, a maximum error of 0.01 in. corresponds to an angular error of 12′. Conversely, if angles are to be plotted within a maximum error of 12′, the protractor should be at least 6 in. in diameter. Likewise, if angles are to be plotted within a maximum error of 6′ of arc, the protractor should be at least 12 in. in diameter. Hence, for high precision, such as is usually desired in plotting control lines, the protractor must be of such a size that it becomes unwieldy and the more convenient method of tangents is used. However, if much work of high precision is to be done, the protractor method is the more rapid, and the expense of an instrument of precision is warranted. Such a protractor is shown in Fig. 14-6. It is made of german silver and is

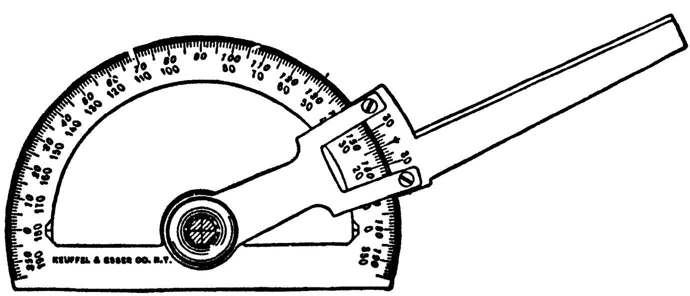


FIG. 14-6. Precise Protractor.

provided with a vernier capable of reading angles to the nearest minute.

The protractor has its greatest usefulness in plotting details. For this work it is usually provided with an arm marked with the proper scale, so that both a distance and an angle can be plotted with one operation. Figure 14-7 represents a simple ring cut from a paper

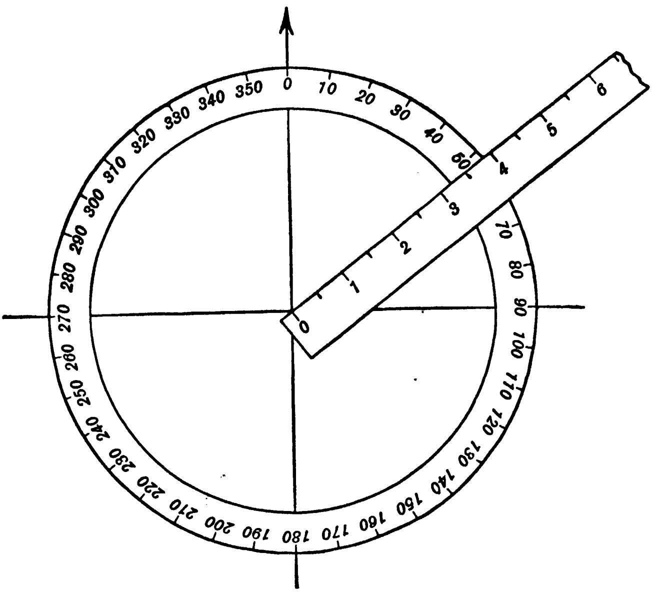


FIG. 14-7. Protractor for Plotting Details.

protractor and suitably numbered. In use it is properly oriented at a control station by means of cardinal direction lines and held in position by thumbtacks or drafting tape. Then any scale can be used as an arm to lay off any given angle and distance in one operation.

The above arrangement permits the plotting of angles of any magnitude and of any distances to the limit of the scale.

The Tangent Method.—The method of tangents makes use of the condition that the numerical value of the tangent of an angle expresses the ratio of the opposite to the adjacent side. Hence, if the adjacent side is given some basic length, as 10 in., the length of the opposite side will be equal to the numerical value of the tangent of

a given angle (in a table of tangents) multiplied by 10 in. Thus in Fig. 14-8, tan $BAD = BD/AD$. Then, if angle $BAD = 30°$, and if adjacent side AD is made 10 in., tan $30° = 0.577 = BD/10$ in. and $BD = 5.77$ in. Likewise, if $CAD = 20°$, tan $20° = 0.364$, and $CD =$

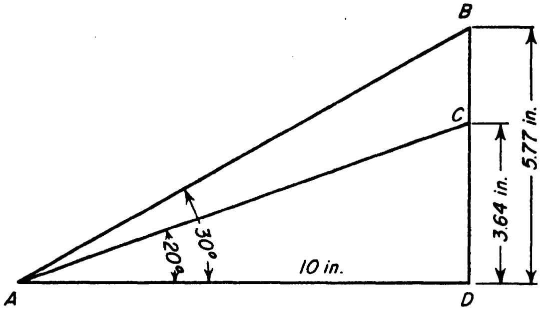


FIG. 14-8. Tangent Method of Plotting Angles.

3.64 in. Thus, any angle may be laid off from side AD at vertex A.

Figure 14-9 illustrates the method of plotting a traverse, $ABCD$, by this method. At B the deflection angle $25°15'$ R is laid off from side AB extended as a base. The length of side BC is scaled to locate

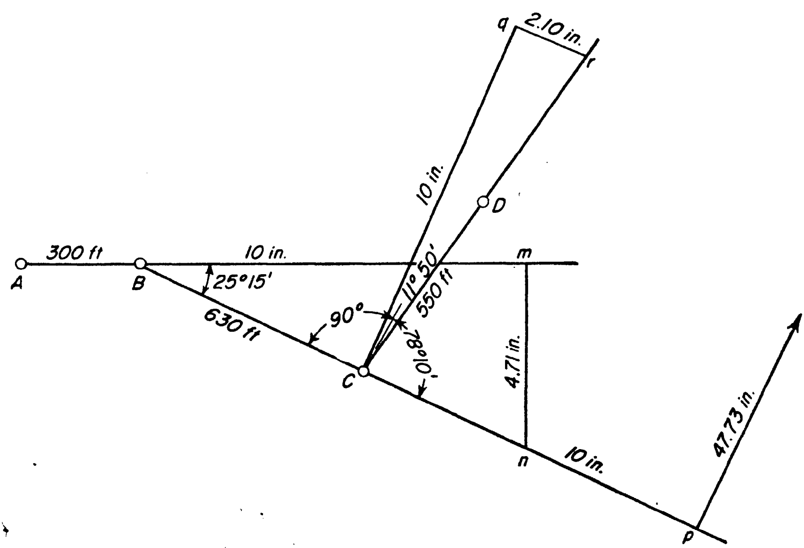


FIG. 14-9. Tangent Method of Plotting a Traverse.

station C. Here the deflection angle $78°10' L$ is to be plotted. It is evident that if base Cp is used, the perpendicular would extend beyond the edge of the paper; hence, the complement of the angle, $11°50'$, is layed off from base Cq, drawn perpendicular to Cp. The direction Cr is thus fixed and the distance CD is scaled thereon.

It is evident that if a base of 10 in. is used, an accuracy is secured equivalent to that provided by a protractor 20 in. in diameter.

14-10. Plotting Traverses Obviously a traverse may be plotted by laying off the angles by either of the methods previously described and by scaling the proper distances between stations. The data for traverses are provided in various forms, depending on the methods used in the field, i.e., deflection angles, interior angles, azimuths, or angles-to-right. The method of plotting deflection angles has been described above, and since deflection angles provide a convenient method of plotting, it is common practice to calculate these angles for use in plotting, when either of the other methods of traversing has been used.

Bearings or azimuths, however, may be plotted directly. By this method the reference meridian is drawn through each vertex, from which the bearing or azimuth angle is laid off either with a protractor or by the tangent method.

It is assumed that mistakes and important errors in the field work have been eliminated before plotting work begins. In the case of a closed traverse, the error of closure is balanced by one of the methods previously described (see Art. 7-7).

14-11. Method of Coordinates Another method of plotting is that which makes use of coordinates, for which the computations have been explained in Art. 7-8.

The first steps in plotting the coordinates is to construct a rectangle whose dimensions are the maximum latitude and the maximum departure in the table of total coordinates. This rectangle will then just enclose the traverse, and the west and south sides constitute the axes of coordinates. This rectangle should be accurately constructed, and its construction can be checked by scaling the two diagonals, which should be exactly equal in length.

Having constructed the enclosed rectangle, the latitude ordinates of all points are scaled off along both the north-south sides, and the departure abscissas are scaled along the east-west sides (see Fig.

14-10). A straightedge is then used to find the intersections of the lines which fix the positions of all points in the traverse.

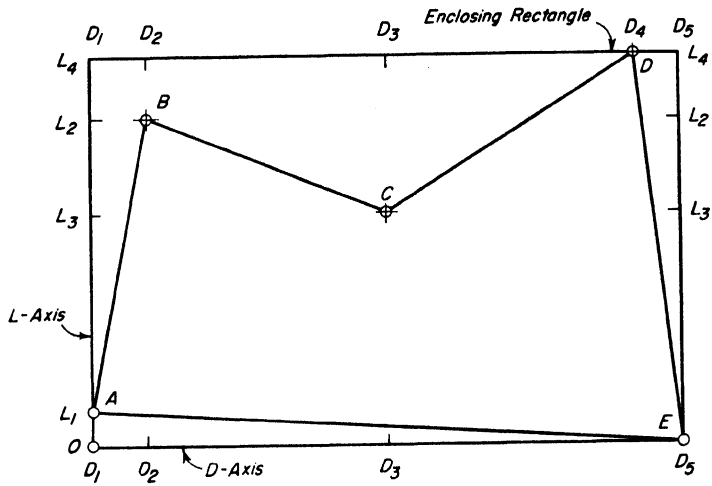


FIG. 14-10. Plotting by Coordinates.

14-12. Checks The plotting of all important traverses should be checked. If a traverse is closed, of course the closing error will afford a check on the work, but, in the case of an open traverse, it will be necessary to check both the angles and the distances. The checking of traverses plotted by the method of coordinates will be considered separately.

1. *Angles.* In checking the angles of a traverse, one of two procedures will be followed, depending on whether a protractor or the method of tangents has been used.

Let *ABCDEFGH*, Fig. 14-11, represent an open traverse. If the deflection angles have been plotted with a protractor, the best method of checking the angles at intervals is that of a *cut-back,* by which a given course is produced backward to intersect a previous course. This angle of intersection is calculated from the given angles and then is measured with the protractor. If the measured and calculated angles agree, all intervening angles are thus checked. For example, course *GH* is produced back to intersect course *AB* extended. The calculated angle of intersection is 11°, then the plotting of angles *B, C, D, E, F,* and *G* is verified.

Angles that have been plotted by the method of tangents can be checked for gross mistakes by the use of a protractor. A better check is the cut-back described above. If this latter method is not convenient, a good check is provided by the use of a different base. Thus,

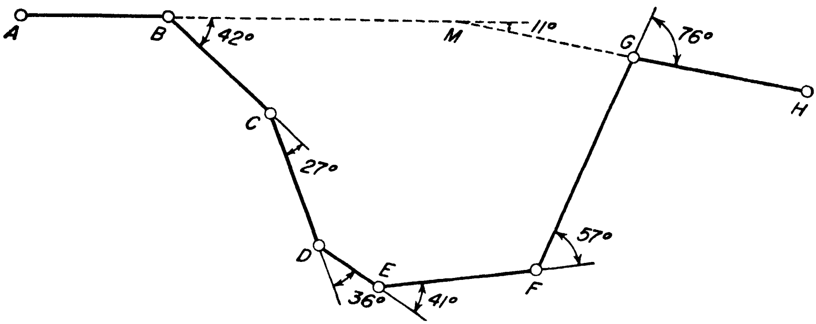


FIG. 14-11. Checking an Open Traverse.

if the original work is done with a base of 10 in., it may be checked quickly and fairly accurately by the use of a base of 5 in.

2. *Distances* are checked by using a different scale and multiplying the plotted dimension by the proper scale ratio. Thus, if a scale of 50 divisions to the inch has been used in the original work, it may be checked by using a scale of 40 divisions to the inch and multiplying the plotted dimensions by 5/4. Another method is to take a strip of paper and mark on its edge the successive lengths of the traverse courses as *AB*, *BC*, etc. The total length so obtained is scaled and should equal the total given length of the traverse courses included.

3. *Coordinates*. In the case of coordinates each station is plotted independently of the others, and no indicated errors of closure are afforded. Accordingly, each angle and distance must be checked separately and the cut-back method cannot be used.

A rough check on both the angles and the distances is afforded by scaling the length of each course. But, if a close check is desired, it may be secured by checking the angles and distances separately as has been described in the preceding paragraphs.

14-13. Distribution of Plotting Errors If the errors of plotting the traverse are appreciable, they are adjusted on the map, as will now be explained. Consider the closed traverse *ABCDEA*, shown in Fig. 14-12. The plotted final point falls at *F* instead of *A*, and dis-

tance FA represents the errors of plotting the traverse, disregarding the errors of the field measurements. If this error is appreciable, it is adjusted along the traverse by drawing the closing line FA; then parallel lines are drawn from each vertex of the traverse, as BB',

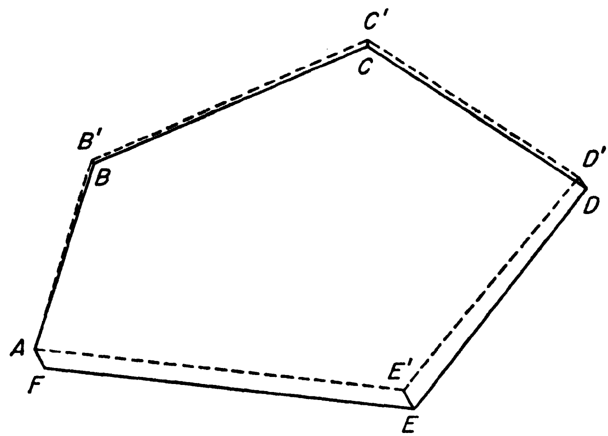


FIG. 14-12. Distributing Plotting Errors.

CC', etc. The adjusted distance BB' is proportional to the total error FA, as the length of the course AB is to the total perimeter. Likewise, the adjusted distance CC' is proportional to FA as distance A-C is to the perimeter. In like manner, each adjusted distance is found and the adjusted positions of all stations are connected by the dashed lines, thus indicating the shape of the adjusted traverse.

In the case of an open traverse, no adjustment is afforded unless the position of the terminus is known from other sources. For this latter condition, the closing error and the adjusted position of all stations are found as indicated above.

14-14. Methods of Plotting Compared The protractor method is most rapid and convenient for plotting angles, but unless the best instruments and office facilities are available, the method is not as accurate as the method of tangents or coordinates. The protractor method is universally used for plotting details.

The method of tangents is accurate and is well adapted to plotting the deflection angles of route surveys. It requires some computations and is slower than the protractor method.

The method of coordinates is both rapid and accurate but the computations are considerable. However, if the coordinates have already been computed as a check on the field work, this method is excellent. Since each point is plotted independently of others, the errors are not cumulative, which is an important advantage. However, this advantage is offset somewhat by the condition that no closing error is afforded to provide a check on the work as a whole.

As between bearings and deflection angles, the former are not subject to cumulative errors as are the latter; hence, in extensive work, the resultant errors are likely to be less with bearings than with deflection angles.

Office Problem

Below are given the field data of a topographic survey of an estate. Make the proper reductions and plot the map, using the methods described in previous chapters.

TRAVERSE

Course	*Distance*	*Azimuth*
AB	449.4	305°02′
BC	397.2	55°49′
CD	507.8	107°38′
DA	551.2	233°28′

DETAILS

Inst. at Station A H.I. = 4.2 Elev. 842.9

K = 100, $(F + C)$ neglect

Sta.	*Az.*	*Rod*	*Angle*	*Remarks*
B	305°02′			
1	248°30′	0.25	(4.6)	East line of 20-ft drive, on line with N highway fence.
2	267°35′	2.16	−1°16′	Fence corner
3	306°10′	1.20		SW corner of house, 40 × 30 ft
4	320°10′	0.90	+1°28′	SE corner of house
5	10°35′	1.38	−1°25′	SW corner of barn, 30 × 60 ft
6	30°30′	1.60	−1°28′	SE corner of barn
7	94°25′	1.34	(4.8)	Fence corner
8	100°35′	0.55	(4.0)	On north fence line of a road 60 ft wide.
B	305°01′	check		

The area within the enclosure about the house and barn is in grass; the area north and west of the enclosure is cultivated; north and east is pasture, except that north of the stream is a wood lot.

Inst. at Station B H.I. = 4.2 Elev. 844.7

Sta.	*Az.*	*Rod*	*Angle*	*Remarks*
C	55°49′			
1	111°35′	3.02	−0°37′	NW corner of shed 20 × 20 ft
2	120°05′	3.16	(4.3)	NW corner of house
3	150°00′	3.07	−1°15′	Fence corner
4	201°25′	3.18		Center of road, 60 ft wide
5	203°50′	2.90	−2°28′	Fence corner
6	318°20′	1.72	(0.7)	Point on fence line
7	342°35′	3.82	(3.7)	Fence corner
C	55°50′	check		

Inst. at Station C H.I. = 4.0 Elev. 831.1

Sta.	*Az.*	*Rod*	*Angle*	*Remarks*
B	235°49′			
1	5°35′	1.42	(8.5)	Fence corner
2	61°00′	2.91	−2°20′	On north fence in creek
3	149°00′	3.38	+0°50′	Fence corner
4	177°15′	2.90	+1°34′	On fence at end of 20 ft drive
5	211°25′	3.40	+1°45′	Fence corner
6	287°20′	4.65	+1°46′	Fence corner
B	235°47′	check		Pts. C_1 and C_4 are connected by a fence.

Inst. at Station D H.I. = 4.2 Elev. 822.8

Sta.	*Az.*	*Rod*	*Angle*	*Remarks*
A	233°28′			
1	28°35′	3.36	+1°36′	Fence corner
2	33°05′	3.52		Centerline of N and S road
3	105°50′	1.68	(5.8)	On east fence in creek
4	152°30′	4.15		Center of cross roads
5	154°30′	3.74	+0°50′	Fence corner
6	215°00′	0.44	(6.2)	Bend in creek
7	246°05′	3.38	+2°12′	Fence corner
A	233°30′	check		

The map should be finished in ink and colors, and a suitable title supplied. The scale suggested is 1 in. = 100 ft, and the contour interval, 2 ft.

If desired, the elevations may be disregarded, and the map drawn and finished without contour lines.

CHAPTER 15

DETERMINATION OF A TRUE MERIDIAN

15-1. Remarks The astronomic determination of a true meridian is still frequently required of the civil engineer and surveyor. The true bearing of a line can be determined with various grades of accuracy by both stellar and solar observations and with a variety of instruments. The following treatment, however, is confined to observations with an ordinary engineer's transit upon the North Star, Polaris, for the purpose of securing an azimuth which will be correct within one minute of arc.

Satisfactory azimuth control for many surveys can be provided with comparative ease through the application of the principles and methods which are explained in the following sections. It will be demonstrated that, with only a rudimentary knowledge of astronomy and the use of conventional equipment, the surveyor has a means for checking the azimuths of lines of extended open traverse, ascertaining the true bearings of land boundaries, and for expressing directions in terms of azimuths or bearings which are unchanging. Furthermore, as the advanced student of surveying learns, the use of state plane coordinates frequently makes necessary an astronomic determination of the true azimuth of a line before the state grid azimuth can be calculated.

15-2. Celestial Sphere The procedures of practical field astronomy are greatly simplified and an understanding of space relationships more easily achieved by the concept of the celestial sphere. The stars, which are at astronomical distances from the earth, are considered to be fixed on the inner surface of a sphere of infinite radius whose center is the earth. It is immaterial in this discussion

whether the observer's position on the earth's surface, or the center of the earth, is considered as the center of the celestial sphere, since the radius of the earth is utterly insignificant when compared with the distances to the stars.

Figure 15-1 depicts the celestial sphere whose essential elements are defined as follows:

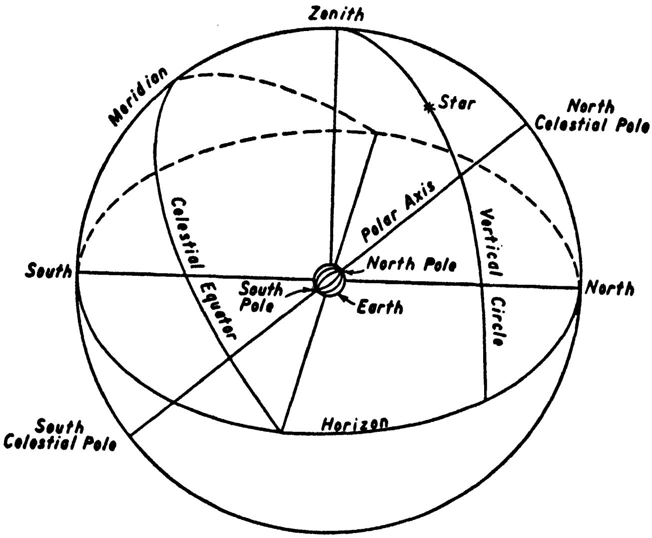


FIG. 15-1. Celestial Sphere.

Great Circle.—A great circle of a sphere is the trace on its surface of the intersection of a plane passing through the center of the sphere.

Celestial Poles.—The north and south geographic poles are at the extremities of the polar axis of the earth. The celestial poles are situated at the points where the earth's polar axis produced intersects the celestial sphere.

Celestial Equator.—The celestial equator is a great circle whose plane is perpendicular to the celestial polar axis.

Zenith.—The zenith is that point on the celestial sphere where a plumb line extended upward intersects it. The opposite point is termed the nadir.

Horizon.—The horizon is the great circle which is situated half-

way between the observer's zenith and nadir points and whose plane is perpendicular to the plumb line. Azimuth is measured in the plane of the horizon.

Vertical Circle.—A vertical circle is the great circle passing through the observer's zenith and any celestial object. Obviously it must be perpendicular to the horizon. The altitude of a heavenly body above the horizon is measured along a vertical circle.

Hour Circle.—An hour circle is the great circle joining the celestial poles and passing through any celestial body. Thus, there is an hour circle for each object in the sky. Of necessity, an hour circle must be perpendicular to the celestial equator.

Meridian.—The observer's celestial meridian is both an hour circle and a vertical circle, since it passes through the celestial poles as well as the zenith and nadir points.

It is earnestly recommended that the student fix well in mind the foregoing definitions, because a clear understanding of them will be most helpful in later discussions.

15-3. Apparent Motion of Celestial Sphere The earth is a planet which completes a rotation from west to east about its polar axis in approximately 24 hours and effects a circuit about the sun in approximately 365 days.

For the purposes of astronomical observations, the earth is regarded as stationary and the celestial sphere is assumed to rotate about it from east to west. This concept of motion of the celestial sphere will be helpful in understanding the apparent movement from east to west of the celestial bodies and particularly in making clear the apparent daily rotation of the pole star, Polaris.

15-4. Time Time and the conversion of one kind of time to another are of great significance in all aspects of astronomy, since the daily motion of the stars and their positions at any instant are intimately related to time. Although several kinds of time are used in astronomy, it will be necessary to deal only with the familiar standard time in this abbreviated treatment of the azimuth problem.

If the movement of the sun* from east to west is considered, it is recognized that its crossing of each meridian signalizes noon for all points on that meridian and that meridians to the east have

* Here the informed reader may wish to make the proper distinction between the true sun and the fictitious body which is termed the mean sun.

already experienced noon while those to the west have yet to experience noon.

Figure 15-2 shows the standard time zones that have been established in the United States. It will be noticed that the central meridians of the time belts differ by 15° of longitude, or 1 hour of time,

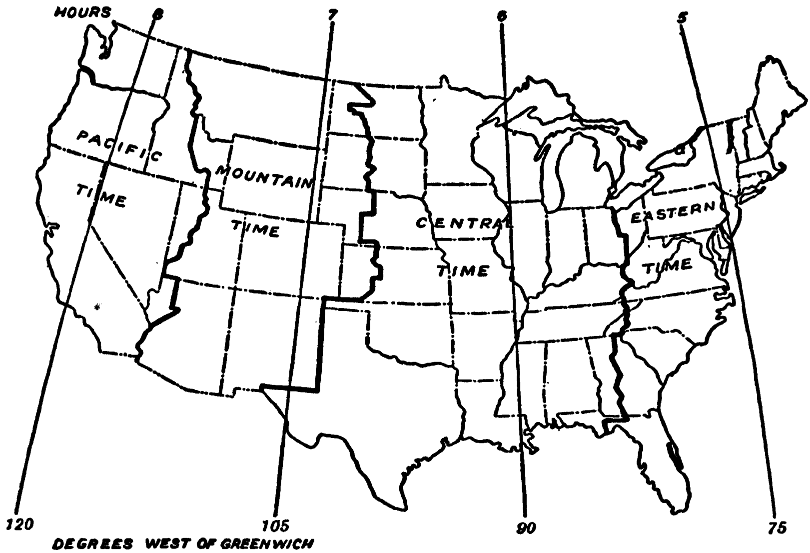


FIG. 15-2. Standard Time in the United States. Eastern Standard Time is the local civil time at the meridian 5 hours west of Greenwich; it is therefore 5 hours earlier than Greenwich Civil Time. Central, Mountain, and Pacific Times are respectively 6, 7, and 8 hours earlier than Greenwich Civil Time.

since the sun completes an apparent revolution about the earth every 24 hours. Although the instant of noon (12 o'clock) on any observer's meridian is nominally that corresponding to the appearance of the sun exactly over the meridian, it is obvious that the use of a multiplicity of local times would result in endless confusion. Hence, all timepieces within a given time zone are set to keep the same kind of time as that which pertains to the zone's central meridian only. This kind of time is termed *standard time*.

Since the sun apparently moves from east to west, a given moment such as noon is experienced earlier in the zones to the east. For example, 12 o'clock Central Standard Time (CST) is 1 P.M. Eastern

Standard Time (EST) and 11 A.M. Mountain Standard Time (MST).

The prime meridian for reckoning world longitude passes through the observatory at Greenwich, England. Thus the longitude of the central meridian of the Greenwich standard time belt is 0° or 0^h. The standard time for this zone is called Greenwich Civil Time (GCT) (also Greenwich Mean Time or Universal Time) and is widely used in astronomical work. GCT is obtained merely by adding to the standard time of any zone in the United States the appropriate number of hours of longitude that its central meridian is removed from Greenwich.

Daylight Time in any zone is equivalent to standard time in the zone next removed to the east; for example, 9 A.M. Central Daylight Time (CDT) is the same instant as 9 A.M. EST.

In astronomical calculations the hours of the day are numbered consecutively from 0 to 24 beginning at midnight. Thus an event which occurs at 3:00 A.M., September 1, would be recorded as 3^h00^m September 1, and an event 12 hours later would be recorded as 15^h00^m September 1. The designations of A.M. and P.M. are unnecessary and are frequently productive of serious errors in the expression of time.

EXAMPLE: Find the GCT of an observation made at 9:50 P.M., CST.

$$\begin{array}{rl} & 9{:}50 \quad \text{P.M., CST} \\ + & \underline{12 \qquad\qquad\qquad} \\ & 21^h50^m \ \text{CST} \\ + & \underline{\ 6 \qquad\qquad\qquad} \\ & 3^h50^m \ \text{GCT(on following day)} \end{array}$$

15-5. Time Signal Service The ease with which correct standard time, or the error of one's timepiece, can be obtained has contributed materially to the simplicity with which the results of astronomic observations for azimuth can be calculated. Although extreme accuracy in time is not required if azimuth values are to be correct to the nearest minute only of arc, it is recommended as a principle of uniformity that any of the following time services be utilized to secure the watch correction to the nearest second.

If an ordinary short-wave radio set is available, time signals can be received 24 hours a day from Station WWV, the National Bureau

of Standards' Central Propagation Laboratory in Washington, D.C. Every second there is a tick and every five minutes there is voice identification of the signals. These time signals are transmitted on a frequency band ranging from 2.5 to 25 megacycles.

Correct time can be obtained also from Western Union offices.

15-6. Latitude and Longitude The latitude and longitude of the observer furnish an expression of his position on the earth. Since values of these coordinates are required for astronomic computations of azimuth, it is desirable to define them and provide information as to how they can be obtained.

Latitude is merely the angle between the direction of the plumb line and the plane of the earth's equator. It can also be defined as the angular distance that the observer is north or south of the equator.

Longitude is the angular distance that the observer is either east or west, 0°-180°, from the Greenwich meridian. Longtitude can be expressed either in terms of arc or time measure. Thus, the longitude of a point in Urbana, Illinois, is $5^h52^m54^s$ West or 88°13′30″ West. Note that a careful distinction must be maintained between arc and time expressions of the same longitude because a minute of arc is not equivalent to a minute of time. The following tabulation offers a useful summary of time and arc relations:

$$
\begin{array}{rclcrcl}
24^h & = & 360^\circ & \qquad & 1^\circ & = & 4^m \\
1^h & = & 15^\circ & & 1' & = & 4^s \\
1^m & = & 15' & & & & \\
1^s & = & 15'' & & & &
\end{array}
$$

Both latitude and longitude can be scaled from a good map. The maps or charts published by various government organizations are best for this purpose. Those most widely available are the standard quadrangle topographic maps of the U.S. Geological Survey. Also, some highway maps are now being published with geographic grids. If no reputable map is available, an inquiry for the observer's latitude and longitude may be addressed to the Director, U.S. Coast and Geodetic Survey, Washington, D.C. The location of the observer can be expressed by giving the name of the nearest post office, or in sectionized areas by the number of the section as well as the designation of township.

A simple astronomic procedure for ascertaining latitude is described in the last part of this chapter.

15-7. Position of a Celestial Body In much the same manner that the geographical coordinates, latitude and longitude, are employed to define the position of an observer on the earth, astronomical coordinates are used to indicate the position of a heavenly body on the celestial sphere. These coordinates are Greenwich Hour Angle (GHA) and Declination.

The GHA of a star is the distance, measured westward, from the Greenwich meridian to the hour circle passing through the star. The GHA can be expressed in time measure (0^h to 24^h) or in arc measure (0° to 360°). Since the motion of the celestial sphere is from east to west, while the earth remains stationary, it is apparent that the GHA of all stars constantly increases with time. This rate of increase is somewhat more than 15° per hour of time, because the celestial sphere actually completes slightly more than one complete rotation every 24 hours.

The declination of a star is the angular distance it is above or below the celestial equator. For Polaris, it is convenient to use the term, *polar distance,* which is the angular distance the star is from the North Pole.

Figure 15-3 shows, for a given moment, the GHA and declination

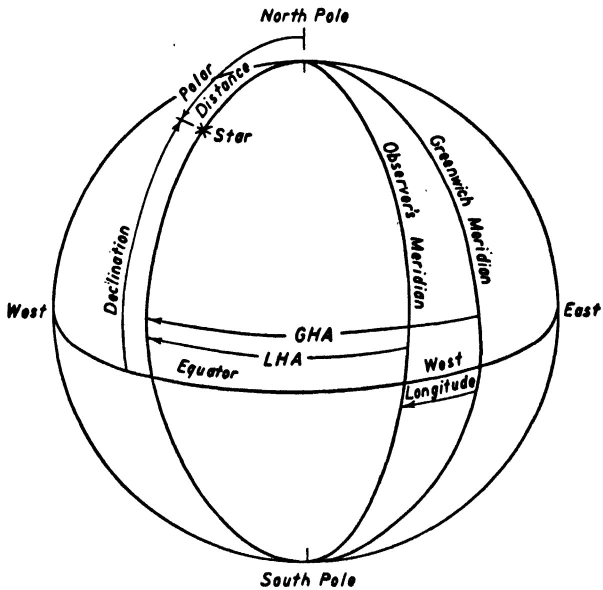


FIG. 15-3. Coordinates of a Star.

of a star. Several publications of astronomic data have tabulations of GHA for certain of the heavenly bodies at the moment of 0^h GCT (Greenwich midnight) for each day in the year. This makes the problem of finding GHA relatively easy. It is merely necessary to convert the standard time of observation to GCT by adding the appropriate number of whole hours (the zone correction) to standard time; then, to the tabulated GHA of the star at 0^h GCT is added the increase in the GHA occurring in the elapsed time interval between 0^h GCT and the GCT of the observation.

TABLE I GREENWICH HOUR ANGLE OF POLARIS

for 0^h Greenwich Civil Time

1954

Day	Jan.	Feb.	March	April	May	June
	° ′	° ′	° ′	° ′	° ′	° ′
1	72 07.6	102 50.5	130 34.4	161 12.7	190 46.3	221 13.8
2	73 07.0	103 50.0	131 33.8	162 12.0	191 45.4	222 12.7
3	74 06.4	104 49.4	132 33.2	163 11.2	192 44.4	223 11.5
4	75 05.9	105 48.9	133 32.5	164 10.4	193 43.4	224 10.4
5	76 05.3	106 48.3	134 31.9	165 09.5	194 42.4	225 09.3
6	77 04.7	107 47.8	135 31.3	166 08.7	195 41.4	226 08.1
7	78 04.2	108 47.2	136 30.6	167 07.9	196 40.4	227 07.0
8	79 03.6	109 46.7	137 30.0	168 07.1	197 39.4	228 05.8
9	80 03.0	110 46.1	138 29.3	169 06.2	198 38.4	229 04.7
10	81 02.5	111 45.5	139 28.6	170 05.4	199 37.4	230 03.5
11	82 01.9	112 45.0	140 28.0	171 04.6	200 36.4	231 02.4
12	83 01.4	113 44.4	141 27.3	172 03.7	201 35.3	232 01.2
13	84 00.8	114 43.9	142 26.6	173 02.9	202 34.3	233 00.1
14	85 00.3	115 43.3	143 26.0	174 02.0	203 33.3	233 58.9
15	85 59.7	116 42.7	144 25.3	175 01.1	204 32.3	234 57.7
16	86 59.2	117 42.2	145 24.6	176 00.2	205 31.2	235 56.6
17	87 58.6	118 41.6	146 23.9	176 59.3	206 30.1	236 55.4
18	88 58.1	119 41.0	147 23.2	177 58.5	207 29.0	237 54.2
19	89 57.5	120 40.4	148 22.5	178 57.6	208 28.0	238 53.1
20	90 57.0	121 39.8	149 21.8	179 65.7	209 26.9	239 51.9
21	91 56.5	122 39.2	150 21.0	180 55.8	210 25.8	240 50.7
22	92 55.9	123 38.6	151 20.3	181 54.8	211 24.8	241 49.5
23	93 55.4	124 38.0	152 19.6	182 53.9	212 23.7	242 48.3
24	94 54.9	125 37.5	153 18.8	183 53.0	213 22.6	243 47.1
25	95 54.3	126 36.9	154 18.1	184 52.1	214 21.5	244 46.0
26	96 53.8	127 36.3	155 17.4	185 51.1	215 20.4	245 44.8
27	97 53.2	128 35.7	156 16.6	186 50.2	216 19.3	246 43.6
28	98 52.7	129 35.0	157 15.8	187 49.2	217 18.2	247 42.4
29	99 52.1		158 15.1	188 48.3	218 17.1	248 41.2
30	100 51.6		159 14.3	189 47.3	219 16.0	249 40.0
31	101 51.1		160 13.5		220 14.9	

EXAMPLE: An observation was made on Polaris at Pittsburgh, Pennsylvania, at $22^h30^m10^s$ EST on June 17, 1954. What is the GHA of Polaris?

$$
\begin{array}{rl}
 & 22^h30^m10^s \text{ EST} \\
+ & \underline{5^h\qquad\quad} \\
 & 27^h30^m10^s
\end{array}
$$

or $3^h30^m10^s$ GCT, June 18.

$$
\begin{array}{rl}
 & 237°54.2' \quad \text{GHA, } 0^h \text{ GCT June 18 (Table I)} \\
+ & \underline{\ \ 52°41.1'} \quad \text{Increase for elapsed time (Table II)} \\
 & 290°35.3' \quad \text{GHA of Polaris at moment of observation}
\end{array}
$$

15-8. Local Hour Angle The Local Hour Angle (LHA) of a star is the distance, measured westward, from the observer's meridian to the hour circle through the heavenly body. LHA can be expressed either in time units (0^h to 24^h) or arc measure (0° to 360°). LHA is obtained from GHA merely by subtracting from it the west longitude or adding the east longitude of the local meridian depending upon the position of the observer. Thus,

TABLE II INCREASE IN GHA FOR ELAPSED TIME SINCE 0^h GCT

Hrs.	Corr.	Min.	Corr.	Min.	Corr.	Sec.	Corr.	Sec.	Corr.
	° ′		° ′		° ′		′		′
1	15 02.5	1	0 15.0	31	7 46.3	1	0.3	31	7.8
2	30 04.9	2	0 30.1	32	8 01.3	2	0.5	32	8.0
3	45 07.4	3	0 45.1	33	8 16.4	3	0.8	33	8.3
4	60 09.9	4	1 00.2	34	8 31.4	4	1.0	34	8.5
5	75 12.3	5	1 15.2	35	8 46.4	5	1.3	35	8.8
6	90 14.8	6	1 30.2	36	9 01.5	6	1.5	36	9.0
8	105 17.2	7	1 45.3	37	9 16.5	7	1.8	37	9.3
7	120 19.7	8	2 00.3	38	9 31.6	8	2.0	38	9.5
9	135 22.2	9	2 15.4	39	9 46.6	9	2.3	39	9.8
10	150 24.6	10	2 30.4	40	10 01.6	10	2.5	40	10.0
11	165 27.1	11	2 45.5	41	10 16.7	11	2.8	41	10.3
12	180 29.6	12	3 00.5	42	10 31.7	12	3.0	42	10.5
13	195 32.0	13	3 15.5	43	10 46.8	13	3.3	43	10.8
14	210 34.5	14	3 30.6	44	11 01.8	14	3.5	44	11.0
15	225 37.0	15	3 45.6	45	11 16.8	15	3.8	45	11.3
16	240 39.4	16	4 00.7	46	11 31.9	16	4.0	46	11.5
17	255 41.9	17	4 15.7	47	11 46.9	17	4.3	47	11.8
18	270 44.4	18	4 30.7	48	12 02.0	18	4.5	48	12.0
19	285 46.8	19	4 45.8	49	12 12.0	19	4.8	49	12.3
20	300 49.3	20	5 00.8	50	12 32.1	20	5.0	50	12.5
21	315 51.7	21	5 15.9	51	12 47.1	21	5.3	51	12.8
22	330 54.2	22	5 30.9	52	13 02.1	22	5.5	52	13.0
23	345 56.7	23	5 45.9	53	13 17.2	23	5.8	53	13.3
24	360 59.1	24	6 01.0	54	13 32.2	24	6.0	54	13.5
		25	6 16.0	55	13 47.3	25	6.3	55	13.8
		26	6 31.1	56	14 02.3	26	6.5	56	14.0
		27	6 46.1	57	14 17.3	27	6.8	57	14.3
		28	7 01.1	58	14 32.4	28	7.0	58	14.5
		29	7 16.2	59	14 47.4	29	7.3	59	14.8
		30	7 31.2	60	15 02.5	30	7.5	60	15.0

$$\text{LHA} = \text{GHA} - \text{west longitude} \tag{15-1}$$

or

$$\text{LHA} = \text{GHA} + \text{east longitude} \tag{15-2}$$

Under certain circumstances, it will be necessary to add 360° (or 24^h) to the GHA of Eq. (15-1) in order to perform the required subtraction.

In connection with azimuth observation on Polaris, it is extremely important to realize that, when the LHA is 0° to 180° (0^h to 12^h), the star is west of north; and when the LHA is 180° to 360° (12^h to 24^h), it is east of north.

A convenient expression of the hour angle position of a star is the meridian angle (t). It is reckoned both westward and eastward from the observer's meridian up to a maximum value of 180°. Thus, when LHA is less than 180°, t is numerically equal to LHA and has the suffix "west." When LHA is greater than 180°, t is equal to 360° less LHA and is termed the meridian angle "east."

For example,

$$\text{When LHA} = 37°10' \text{ then } t = 37°10'\text{W}$$

$$\text{When LHA} = 210°35' \text{ then } t = 149°25'\text{E}$$

15-9. The Astronomical Triangle A spherical triangle is the figure formed by joining any three points on the surface of a sphere by arcs of great circles. A particular spherical triangle having as its vertices a celestial pole, the observer's zenith, and a heavenly body is called the "astronomical triangle," and it is of much importance in engineering astronomy.

Figure 15-4 shows the PZS (pole, zenith, star) triangle as viewed from the observer's zenith. In the typical azimuth problem, the known quantities in the triangle are sides PZ (90° − latitude) and PS (90° − declination), and the included angle, t, at the pole. Side ZS equals (90° − altitude). It is frequently called the *zenith distance.* Through application of the spherical trigonometry, the required angle at Z can be calculated. This angle is commonly termed the bearing angle of the star and is reckoned both east and west from north. In Fig. 15-4 the star is depicted west of the meridian. It will be noticed that t is west and is numerically the same as LHA. If the star were east of the meridian, LHA would exceed 180°, t would become east, and the angle at Z would become directly the azimuth of the star.

15-10. Polaris Polaris is a celestial body of great importance to the civil engineer and surveyor. It is a fairly bright star (magnitude 2.1) situated about 1° from the north celestial pole. It can be easily identified by prolonging the line connecting the two stars in

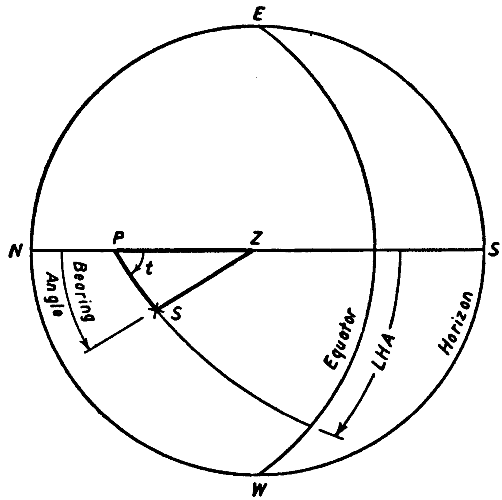


FIG. 15-4. The Astronomical Triangle.

the bowl of the Big Dipper on the side most remote from the handle. It is also helpful to remember that the altitude of Polaris will be always within 1° of the observer's latitude and that there are no other stars near Polaris which are likely to be confused with it.

As viewed from a position on earth (Fig. 15-5), Polaris rotates in a counterclockwise direction about the north celestial pole. It makes a complete revolution approximately every 24 hours.

The instant that Polaris crosses the upper branch of the observer's meridian is called time of Upper Culmination (UC). At that moment LHA of Polaris is zero. Approximately 6 hours later Polaris reaches Western Elongation (WE). Polaris passes over the meridian again at Lower Culmination (LC) and attains its most easterly position at Eastern Elongation (EE).

From this description of the motion of Polaris it is apparent that, at the moments of elongation, the star's motion will be nearly vertical and that its bearing will change but slightly with time. At culmination, however, the rate of change of azimuth of the star with respect to time will be a maximum.

The certainty with which Polaris can be identified, its proximity to the north pole, the fact that it is constantly above the horizon, and the ready availability of specially prepared tables to facilitate the solution of the azimuth problem make it a particularly useful astronomical body.

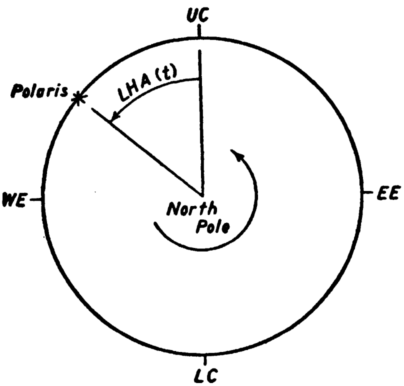


FIG. 15-5. Motion of Polaris (as viewed from the earth).

15-11. The Azimuth Problem The typical azimuth problem consists of two basic operations. They are as follows:

1. The observation of Polaris at any time in order to measure the horizontal angle between the star and a terrestrial signal called the "mark." At the moment a pointing is made on the star, standard time is noted.
2. The computation of the bearing angle of Polaris at the instant of observation. This calculated angle, when appropriately combined with the measured field angle, will yield the required astronomic azimuth or true bearing of the line.

15-12. Field Procedure This treatment emphasizes the inherent advantages of observing Polaris for azimuth at any local hour angle. The facility with which the computations can be made and results of good quality obtained make the local hour angle method an entirely satisfactory procedure. It is preferred to the elongation method even for the most precise azimuth determinations of geodetic surveys.

It can be easily verified that, at the most unfavorable moment, i.e., Polaris at culmination, an error of as much as 1 min. in time causes, at a latitude of 40°, a bearing angle error of only 0.3′ of arc. Furthermore, the local hour angle method permits the observation to be made at any moment convenient to the observer. He is not required to wait for the precalculated instant of elongation which may take place at a most unsuitable time or when the star happens to be obscured by clouds.

The field procedure begins with setting up the transit over the

station, performing the usual centering and leveling, and taking an initial sight at the mark. It will be necessary to illuminate the cross wires by shining a flashlight obliquely into the objective end of the telescope. Optical transits, however, are wired for internal illumination. It is best that the terrestrial line be at least 600 ft long so that the objective lens does not require refocusing when sighting the star. A good signal at the mark consists of a slit cut in the side of a box or tin can in which a light is placed. The signal must be well centered, of course, over the mark. When pointing upon the star, it is very essential that Polaris appear as a tiny pinpoint of light rather than as a disc. Difficulty will be experienced in finding the star unless the objective lens is correctly focused. This can be accomplished by taking a preliminary sight on any other prominent heavenly body or upon a distant terrestrial light.

At the moment the vertical wire is on the star, the watch time is observed and recorded, and then the horizontal angle is read and recorded. The telescope is then inverted, the upper motion reversed, the plate bubbles recentered, and a second pointing is taken on the star. After the time of the second sight is recorded, the mark is again pointed upon and the horizontal circle is read and recorded. This pair of observations constitutes a set. Another set may be taken as a check or to increase the accuracy of the field observations.

15-13. Ephemerides The tables in this chapter are available in more complete form in the ephemerides published by governmental agencies and by the various makers of surveying instruments. All are published annually, and a copy for the current year in which field observations are being conducted is indispensable for making the necessary reduction computations.

The most complete publication is the "American Ephemeris and Nautical Almanac," which is available through purchase from the Government Printing Office, Washington, D.C. A particularly useful, entirely adequate, and very inexpensive publication for the surveyor is the "Ephemeris of the Sun, Polaris, and other Stars," which is prepared by the Bureau of Land Management, Department of the Interior. It also is available from the Government Printing Office. The abridged ephemerides provided by the instrument makers are furnished without charge and will generally provide all the data necessary for the computations treated here.

15-14. Reduction Procedures Three reduction procedures for calculating the azimuth of Polaris at any hour angle will be presented.

1. A formula of wide and general application for computing the precise azimuth of any celestial body is as follows:

$$\tan Z = \frac{\sin t}{\cos \phi \tan \delta - \sin \phi \cos t} \tag{15-3}$$

where ϕ = latitude
δ = declination
t = hour angle
Z = azimuth of star

The quantity Z is reckoned east or west from true north.

2. For a less precise value of the azimuth a more simple formula can be used. It can be derived directly from the law of sines for spherical trigonometry as applied to the *PZS* triangle of Fig. 15-4. Hence, it is seen that

$$\sin Z = \frac{\sin t \sin (90° - \delta)}{\sin (90° - h)}$$

or
$$\sin Z = \frac{\sin t \sin p}{\cos h} \tag{15-4}$$

where p = polar distance, $90° - \delta$.
h = altitude

For a circumpolar star like Polaris, angles Z and p will always be small, Therefore, we can substitute the values of the angles in seconds (or minutes) for their sines and obtain

$$Z = \frac{p \sin t}{\cos h} \tag{15-5}$$

where Z and p are both in seconds or in minutes of arc.

It will be noted that Eq. (15-5) requires the measurement of the altitude, h. This is the value after instrumental corrections and refraction are applied to the observed altitude. If a reliable measurement of altitude seems unlikely and the latitude, upon the other hand, has been well established, the following formula can be used:

$$Z = \frac{p \sin t}{\cos (\phi + p \cos t)} \tag{15-6}$$

The quantity $p \cos t$ can be obtained from prepared tables like Table VI. This quantity is the difference between the altitude of Polaris and the North Pole at any hour angle, t. Note that Table

VI is designed to convert the true altitude of Polaris to the altitude of the pole which latter quantity is numerically the same as the observer's latitude as explained in Art. 15-16. In the use of Eq. (15-6) the quantity $p \cos t$ taken from Table VI is either added to or subtracted from the known latitude, depending upon whether the star is above or below the pole at the moment of observation. The position of Polaris is given by its hour angle as shown in Fig. 15-5.

3. A convenient solution of the *PZS* triangle for the azimuth of Polaris can be made with certain prepared tables. Such a solution is presented in detail in Art. 15-15.

15-15. Azimuth from Polaris at Any Time A typical reduction computation is presented for an azimuth observation made upon Polaris on May 5, 1954, at Urbana, Illinois. The given time and horizontal angle represent the mean values from a single set, direct and reverse, of observations on the star.

The data are as follows:

Station occupied: Illinois	Mark: Sta. 25
Latitude: 40°06.3′ N	Date: May 5, 1954
Longitude: 88°13′30″ W	Observer: J. L. M.
Observed time: $8^h23^m20^s$ p.m., CST	Watch 35^s slow
Angle: mark to star (clockwise) $46°17\frac{1}{2}'$	

It is required to find the true bearing of the line. The solution is as follows:

1. *Calculation of GCT*

Watch time (P.M.)	$8^h23^m20^s$ CST	
Correction	$+\ 35^s$	
Corrected time	$8^h23^m55^s$	
Time (24-hr. basis)	$20^h23^m55^s$	
Zone correction	$+6^h$	
GCT	$2^h23^m55^s$	
Greenwich date	May 6	

2. *Calculation of LHA*

GHA at 0^h GCT	195°41.4′	(Table I)
Increase in GHA	36°04.6′	(Table II)
GHA	231°46.0′	
Less west long.	88°13.5′	
LHA	143°32.5′	
t	143°32.5′ (west)	

TABLE III BEARING OF POLARIS AT ALL LOCAL HOUR ANGLES

1954

For local hour angles 0° to 180° the star is west of north, and from 180° to 360° is it east of north.

Lat.	30°	32°	34°	36°	38°	40°	42°	44°	46°	48°	Lat.
LHA											LHA
	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	
0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	360
5	0 5.8	0 5.9	0 6.0	0 6.2	0 6.4	0 6.5	0 6.8	0 7.0	0 7.2	0 7.5	355
10	0 11.5	0 11.7	0 12.0	0 12.3	0 12.7	0 13.0	0 13.5	0 13.9	0 14.4	0 15.0	350
15	0 17.1	0 17.5	0 17.9	0 18.4	0 18.9	0 19.4	0 20.1	0 20.7	0 21.5	0 22.3	345
20	0 22.6	0 23.1	0 23.7	0 24.3	0 24.9	0 25.7	0 26.5	0 27.4	0 28.4	0 29.5	340
25	0 27.9	0 28.5	0 29.2	0 30.0	0 30.8	0 31.7	0 32.7	0 33.8	0 35.1	0 36.4	335
30	0 33.0	0 33.8	0 34.6	0 35.4	0 36.4	0 37.5	0 38.7	0 40.0	0 41.5	0 43.1	330
35	0 37.9	0 38.7	0 39.6	0 40.6	0 41.7	0 43.0	0 44.3	0 45.8	0 47.5	0 49.4	325
40	0 42.4	0 43.4	0 44.4	0 45.5	0 46.8	0 48.1	0 49.7	0 51.3	0 53.2	0 55.3	320
45	0 46.7	0 47.7	0 48.8	0 50.0	0 51.4	0 52.9	0 54.6	0 56.4	0 58.5	1 0.8	315
50	0 50.5	0 51.6	0 52.8	0 54.2	0 55.6	0 57.3	0 59.1	1 1.1	1 3.3	1 5.7	310
55	0 54.0	0 55.1	0 56.4	0 57.9	0 59.4	1 1.2	1 3.1	1 5.2	1 7.6	1 10.2	305
60	0 57.0	0 58.3	0 59.6	1 1.1	1 2.8	1 4.6	1 6.6	1 8.9	1 11.4	1 14.1	300
65	0 59.6	1 0.9	1 2.3	1 3.9	1 5.6	1 7.5	1 9.6	1 12.0	1 14.6	1 17.5	295
70	1 1.8	1 3.1	1 4.6	1 6.2	1 8.0	1 9.9	1 12.1	1 14.5	1 17.2	1 20.2	290
75	1 3.5	1 4.8	1 6.3	1 8.0	1 9.8	1 11.8	1 14.0	1 16.5	1 19.3	1 22.3	285
80	1 4.6	1 6.0	1 7.5	1 9.2	1 11.1	1 13.1	1 15.4	1 17.9	1 20.7	1 23.8	280
85	1 5.3	1 6.7	1 8.3	1 9.9	1 11.8	1 13.9	1 16.2	1 18.7	1 21.5	1 24.6	275

TABLE III BEARING OF POLARIS AT ALL LOCAL HOUR ANGLES—(*Continued*)

1954

For local hour angles 0° to 180° the star is west of north, and from 180° to 360^h it is east of north.

Lat.	30°	32°	34°	36°	38°	40°	42°	44°	46°	48°	Lat.
LHA											LHA
	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	
90	1 5.5	1 6.9	1 8.5	1 10.1	1 12.0	1 14.1	1 16.4	1 18.9	1 21.7	1 24.8	270
95	1 5.2	1 6.6	1 8.1	1 9.8	1 11.7	1 13.7	1 16.0	1 18.5	1 21.3	1 24.3	265
100	1 4.4	1 5.8	1 7.3	1 8.9	1 10.8	1 12.8	1 15.0	1 17.5	1 20.2	1 23.2	260
105	1 3.1	1 4.5	1 5.9	1 7.5	1 9.3	1 11.3	1 13.5	1 15.9	1 18.6	1 21.5	255
110	1 1.4	1 2.7	1 4.1	1 5.6	1 7.4	1 9.3	1 11.4	1 13.7	1 16.3	1 19.2	250
115	0 59.2	1 0.4	1 1.7	1 3.3	1 4.9	1 6.7	1 8.8	1 11.0	1 13.5	1 16.3	245
120	0 56.5	0 57.7	0 59.0	1 0.4	1 2.0	1 3.7	1 5.6	1 7.8	1 10.1	1 12.8	240
125	0 53.4	0 54.5	0 55.7	0 57.1	0 58.6	1 0.2	1 2.0	1 4.0	1 6.3	1 8.7	235
130	0 49.9	0 50.9	0 52.1	0 53.3	0 54.7	0 56.3	0 57.9	0 59.8	1 1.9	1 4.2	230
135	0 46.0	0 47.0	0 48.0	0 49.2	0 50.5	0 51.9	0 53.4	0 55.2	0 57.1	0 59.2	225
140	0 41.8	0 42.7	0 43.6	0 44.7	0 45.8	0 47.1	0 48.5	0 50.1	0 51.8	0 53.8	220
145	0 37.3	0 38.1	0 38.9	0 39.8	0 40.9	0 42.0	0 43.3	0 44.7	0 46.2	0 47.9	215
150	0 32.5	0 33.2	0 33.9	0 34.7	0 35.6	0 36.6	0 37.7	0 38.9	0 40.3	0 41.7	210
155	0 27.5	0 28.0	0 28.6	0 29.3	0 30.1	0 30.9	0 31.8	0 32.9	0 34.0	0 35.3	205
160	0 22.2	0 22.7	0 23.2	0 23.7	0 24.3	0 25.0	0 25.8	0 26.6	0 27.5	0 28.5	200
165	0 16.8	0 17.2	0 17.5	0 17.9	0 18.4	0 18.9	0 19.5	0 20.1	0 20.8	0 21.6	195
170	0 11.3	0 11.5	0 11.8	0 12.0	0 12.4	0 12.7	0 13.1	0 13.5	0 14.0	0 14.5	190
175	0 5.7	0 5.8	0 5.9	0 6.0	0 6.2	0 6.4	0 6.6	0 6.8	0 7.0	0 7.3	185
180	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	180

This table has been computed for a polar distance of 0°56.8′. For other polar distances the correction from Table V should be applied.

3. *Preliminary Bearing Angle of Polaris.* A double interpolation is made from Table III which is entered with the known latitude and previously calculated LHA. This interpolation effects a mechanical solution of the *PZS* triangle for an average value, 0°56.8′, of the polar distance of Polaris for the year.

LHA	LATITUDE		
	40°	40°06.3′	42°
140°	0°47.1′	0°47.2′	0°48.5′
143°32.5′	—	0°43.6′	—
145°	0°42.0′	0°42.1′	0°43.3′

Thus, the preliminary bearing angle of Polaris is 0°43.6′.

TABLE IV POLAR DISTANCE OF POLARIS
1954

Date	Polar Distance	Date	Polar Distance
	° ′		° ′
January 1	0 56.7	July 1	0 57.2
February 1	0 56.6	August 1	0 57.1
March 1	0 56.7	September 1	0 57.0
April 1	0 56.8	October 1	0 56.9
May 1	0 57.0	November 1	0 56.7
June 1	0 57.1	December 1	0 56.5

Preliminary bearing angle		0°43.6′
Correction (Table V)	+	0.1′
Final Bearing Angle of Polaris		0°43.7′

TABLE V CORRECTIONS TO PRELIMINARY BEARINGS OF POLARIS AS OBTAINED FROM TABLE III

Polar Distance	Bearing					
	0′	20′	40′	1°	1° 20′	1° 40′
° ′	′	′	′	′	′	′
0 57.2	0.0	+0.1	+0.3	+0.4	+0.6	+0.7
0 57.0	0.0	+0.1	+0.1	+0.2	+0.3	+0.4
0 56.8	0.0	0.0	0.0	0.0	0.0	0.0
0 56.6	0.0	−0.1	−0.1	−0.2	−0.3	−0.4
0 56.4	0.0	−0.1	−0.3	−0.4	−0.6	−0.7

4. *Final Bearing Angle of Polaris.* The actual polar distance of Polaris, 0°57.0′, is obtained from Table IV and affords, together with the preliminary bearing angle of Polaris, a means for entering Table V to obtain a supplementary correction. As a glance at Table V discloses, this correction is usually very small.

5. *True Bearing of Line.* The final bearing of Polaris is combined with the measured field angle to obtain the bearing of the line. Since the LHA of Polaris is between 0° and 180°, the star is west of north. A good sketch (Fig. 15-6) depicting the position of the star in relation to the meridian and the terrestrial line always should be drawn.

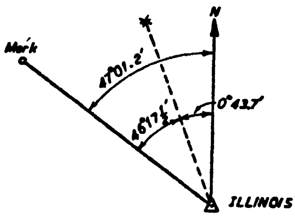


FIG. 15-6. True Bearing of a Line.

The true bearing of the line is thus found to be N 47°01.2′ W.

Alternate solution. An alternate solution of this problem will now be effected with the use of Eq. (15-6) as follows:

$$Z = \frac{p \sin t}{\cos (\phi + p \cos t)} \tag{15-6}$$

where

$$\begin{aligned} p &= 57.0' && \text{(from Table IV)} \\ t &= 143°32.5' && \text{(as previously calculated)} \\ p \cos t &= 45.7' && \text{(from Table VI)} \\ \phi &= 40°06.3' \text{ N.} \end{aligned}$$

hence,

$$Z = \frac{(57.0)(0.59424)}{\cos 39°20.6'} = 0°43.8'.$$

15-16. Latitude from Polaris at Any Time Figure 15-7 portrays a section along the observer's meridian through the portion of the celestial sphere above his horizon. It is readily seen that, since the polar axis is perpendicular to the equator and the plumb line is normal to the plane of the horizon, the altitude of the pole equals the latitude. At culmination, Polaris is above or below the pole by

an amount of arc equal to its polar distance. At elongation, it is directly opposite the pole and its altitude equals the latitude. For

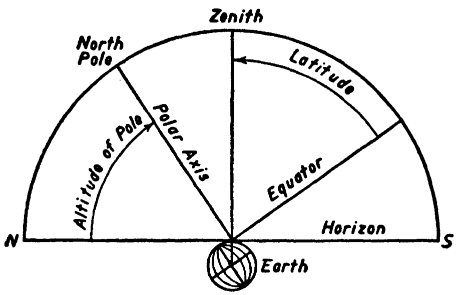


FIG. 15-7. Latitude and Altitude of Pole.

other intermediate positions, Table VI tabulates the angular corrections which, for various values of LHA, can be applied to the observed altitude of Polaris to reduce to the altitude of the pole.

EXAMPLE: An observation for latitude was made upon Polaris at a point in northern Minnesota on July 27, 1954. The observed altitude of Polaris was 48°06½′, and the calculated LHA for the observed moment of observation was 215°20′. What is the observer's latitude?

LHA is 215°20′	or t is 144°40′ E
48°06.5′	observed altitude*
+ 46.4′	Reduction to pole (Table VI.)
48°52.9′	Latitude

Office Problems

15-1. Calculate the GCT corresponding to the following moments of standard time:

(a) 5 P.M. CST Feb. 10 (*Ans.* 23^h00^m Feb. 10)
(b) 23^h50^m EST Mar. 30 (*Ans.* 4^h50^m Mar. 31)
(c) 18^h40^m CDT Jan. 6 (*Ans.* 23^h40^m Jan. 6)
(d) 21^h16^m MST June 2 (*Ans.* 4^h16^m June 3)

* Refraction is neglected. It is actually less than 01′ here.

15-2. For the year 1954 calculate the GHA of Polaris for each of the moments tabulated in Prob. 15-1. (*Ans.* (a) 97°42.2′ (b) 232° 55.5′ (c) 73°03.0′ (d) 287°22.1′.

15-3. If the observer is in west longitude 92°15.0′, calculate both the LHA and t for each of the preceding moments of time. *Ans.*

(a) LHA = 5°27.2′	t = 5°27.2′ W
(b) LHA = 140°40.5′	t = 140°40.5′ W
(c) LHA = 340°48.0′	t = 19°12.0′ E
(d) LHA = 195°07.1′	t = 164°52.9′ E

15-4. An azimuth observation was made upon Polaris on March 17, 1954, at Pittsburgh, Pa. The latitude of the station is 41°40.0′ N, and the longitude is 83°36.5′ W. The mean angle, measured clockwise, from the mark to the star is 23°14½′ and the corresponding observed time is $10^h43^m25^s$ P.M., EST. The watch is known to be 30^s fast. Find the true bearing of the line. (*Ans.* N 24°20.0′ W)

15-5. Suppose that a latitude of 41°35.0′ was erroneously scaled from a map and used in the solution of Prob. 15-4. Make a new calculation of the bearing angle of Polaris using this erroneous value of the latitude. By how much is this bearing angle of the star in error. (*Ans.* 0.1′)

15-6. Suppose the watch correction of Prob. 15-4 were neglected. Make a new calculation of Prob. 15-4 to determine the resulting error in the bearing angle of the star. (*Ans.* 0.1′) (Note that 30^s error in time will have the same effect on the calculation of LHA of Polaris as an error of 7½′ in longitude.)

15-7. A latitude observation was made on Polaris on November 1, 1954, at a point in central California. The observed altitude of Polaris was 37°17′, and the calculated LHA of the star was 289°10′. Neglecting refraction, find the observer's latitude. (*Ans.* 36°59′ N)

TABLE VI CORRECTIONS TO BE APPLIED TO ALTITUDE OF POLARIS TO OBTAIN LATITUDE

1954

t	Cor.	t	Cor.	t	Cor.	t	Cor.
°	′	°	′	°	′	°	′
0	−56.7	45	−39.8	90	+ 0.5	135	+40.3
1	56.7	46	39.1	91	1.5	136	41.0
2	56.6	47	38.4	92	2.4	137	41.7
3	56.6	48	37.7	93	3.4	138	42.3
4	56.5	49	36.9	94	4.4	139	43.0
5	−56.4	50	−36.1	95	+ 5.4	140	+43.6
6	56.3	51	35.4	96	6.4	141	44.2
7	56.2	52	34.6	97	7.4	142	44.8
8	56.1	53	33.8	98	8.3	143	45.4
9	55.9	54	33.0	99	9.3	144	46.0
10	−55.8	55	−32.2	100	+10.3	145	+46.6
11	55.6	56	31.4	101	11.3	146	47.1
12	55.4	57	30.5	102	12.2	147	47.7
13	55.3	58	29.7	103	13.2	148	48.2
14	55.0	59	28.8	104	14.1	149	48.7
15	−54.7	60	−28.0	105	+15.1	150	+49.2
16	54.4	61	27.1	106	16.0	151	49.7
17	54.2	62	26.2	107	17.0	152	50.1
18	53.8	63	25.4	108	17.9	153	50.6
19	53.5	64	24.5	109	18.9	154	51.0
20	−53.2	65	−23.6	110	+19.8	155	+51.4
21	52.9	66	22.7	111	20.7	156	51.8
22	52.5	67	21.7	112	21.6	157	52.2
23	52.1	68	20.8	113	22.5	158	52.6
24	51.7	69	19.9	114	23.4	159	53.0
25	−51.3	70	−19.0	115	+24.3	160	+53.3
26	50.8	71	18.0	116	25.2	161	53.6
27	50.4	72	17.1	117	26.1	162	53.9
28	49.9	73	16.1	118	27.0	163	54.2
29	49.4	74	15.2	119	27.8	164	54.5
30	−49.0	75	−14.2	120	+28.7	165	+54.8
31	48.4	76	13.3	121	29.5	166	55.0
32	47.9	77	12.3	122	30.4	167	55.2
33	47.4	78	11.3	123	31.2	168	55.4
34	46.8	79	10.4	124	32.0	169	55.6
35	−46.3	80	− 9.4	125	+32.8	170	+55.8
36	45.7	81	8.4	126	33.6	171	56.0
37	45.1	82	7.4	127	34.4	172	56.1
38	44.5	83	6.4	128	35.2	173	56.3
39	43.8	84	5.5	129	35.9	174	56.4
40	−43.2	85	− 4.5	130	+36.7	175	+56.5
41	42.6	86	3.5	131	37.5	176	56.5
42	41.9	87	2.5	132	38.2	177	56.6
43	41.2	88	1.5	133	38.9	178	56.6
44	40.5	89	− 0.5	134	39.6	179	56.7
45	39.8	90	+ 0.5	135	40.3	180	56.7

CHAPTER 16

PHOTOGRAMMETRY

16-1. General Remarks The development and application of photography and photogrammetry to surveying and mapping within recent years have constituted an important advance in the art of surveying. This advance has taken place in four distinct steps: (1) the application of photography to surveying from terrestrial views, using the principles of perspective only; (2) the introduction of stereocomparison and stereoscopic vision into the photographic method; (3) the transfer from the terrestrial to the aerial position of the camera; and (4) the development of instruments and methods for utilizing the data in the photographs, both terrestrial and aerial, for surveying and mapping purposes.

Terrestrial photographic surveying came into practical use about 1870 and has had a wide application both in Europe and America. Much of the Rocky Mountain region of the United States, Canada, and Alaska, for example, has been mapped by this method. At the beginning of the present century, stereocomparison came into use, whereby overlapping pictures of the same terrain, viewed simultaneously, permitted a more rapid and practical use of the photographic method. Aerial photography and the methods and instruments recently developed have improved the accuracy and economy of the results obtained to the extent that no important engineering project or survey is completed now without the aid of photogrammetry.

Aerial photogrammetry uses photographs taken from aircraft with the camera axis either vertical or oblique. The resulting views, therefore, are called *vertical* or *oblique* photographs.

In the United States most photogrammetric methods for engineering projects and governmental surveys use vertical photographs almost exclusively, but, in Canada, oblique views are taken exten-

sively. The use of oblique views is advantageous where there is a flat terrain and where a considerable amount of water surface appears in the photographs. An oblique view covers a much greater area than a vertical one, and hence it affords a great economy where conditions are favorable to its use.

Vertical views are taken with the airplane flying a straight course, and with sufficient overlap between adjacent views to afford stereoscopic vision necessary for photogrammetric measurements. The photographs made on a given course constitute a *flight strip,* and a sufficient number of strips are taken to cover a given area. The overlap in the direction of flight is called *forward lap,* and the overlap between pictures in adjacent flights is called *side lap.* The amount of forward lap and side lap is commonly specified as 60 and 30%, respectively.

Aerial photographs are, of course, taken from a rapidly moving aircraft, and hence the position of the camera, either in azimuth or in elevation, is not known. Although the attempt is made to keep the camera axis vertical, a small amount of tilt, the direction of which is unknown is usually present when an exposure is made. Because of these unknown factors, photogrammetric measurements have been beset with many difficulties, but these have been overcome to such an extent that aerial methods have either displaced or considerably modified the ground methods formerly used by all governmental mapping organizations. On practically every important survey or engineering project, by either public or private agencies, aerial photographs are used in some way.

For example, wide use is now being made of photogrammetry in the planning, location and construction of highways. From aerial photographs accurate estimates are made of the costs of the right-of-way, earthwork, and structures by alternate routes, in order to obtain the most feasible and economical route.

16-2. Definitions* *Photogrammetry* is the science and art of obtaining accurate measurements by the use of photography.

Point of View, I, Fig. 16-1, is the center of the camera lens.

Camera Axis is the line passing through the center of the camera lens perpendicular both to the camera plate and the picture plane.

* In the interest of simplicity, some of the definitions used in this chapter are modified somewhat from their correct technical statements. See *Manual of Photogrammetry,* Second Edition, pp. 805–843.

Picture Plane is the plane perpendicular to the camera axis at the focal length distance in front of the lens. It is represented by the positive contact print or *photograph* taken from a plate or film.

Principal Point is the point of intersection, *O*, of the camera axis

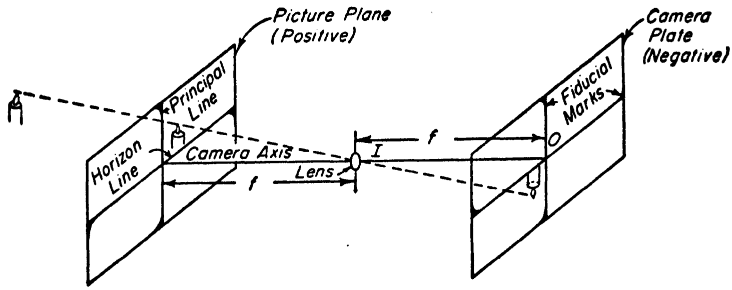


FIG. 16-1. Elements of a Horizontal Photograph.

with either the picture plane (photograph), or the camera plate (negative).

Focal Length, f, is the perpendicular distance from the center of the camera lens to either the picture plane or the camera plate.

Principal Distance. In many photogrammetric procedures the contact prints from original negatives are enlarged, or reduced, or changed because of paper shrinkage before their use in compiling the subsequent maps is begun. In this case the value of *f* (meaning the focal length of the camera) is not applicable to the revised prints. The changed value of *f* is called the *principal distance.* However, it is designated by the letter *f*, and the geometrical relations remain the same as those of the focal length of the camera.

Fiducial Marks are index marks within the camera frame which form images on the edges of the negative. The intersection of straight lines on the photograph connecting these images fixes the principal point of the photograph.

Photograph Nadir is the point of intersection of a vertical (plumb) line through the center of the lens (at the instant of exposure) and the photograph. If there is no tilt of the camera axis when the exposure is made, the photograph nadir and the principal point will be identical.

Ground Nadir is the point of intersection of a vertical line through the center of the lens and the ground surface.

16-3. Perspective Principles of Vertical Photographs Since any photograph is a perspective view, it is subject to the principles of such views whether it is a terrestrial (horizontal) or an aerial (vertical) photograph. The following principles apply to vertical photographs:

1. The photographic images of all vertical lines of objects on the ground will be radial lines which, if extended, will pass through the principal point O.
2. All parallel level lines on the ground, such as the parallel sides of a square tract of level land, will appear as parallel lines on the photograph.

The first principle stated above has the greatest significance in all photogrammetric uses of vertical photographs. Three examples will be mentioned. (1) In Fig. 16-2 the flagpole represents a vertical line

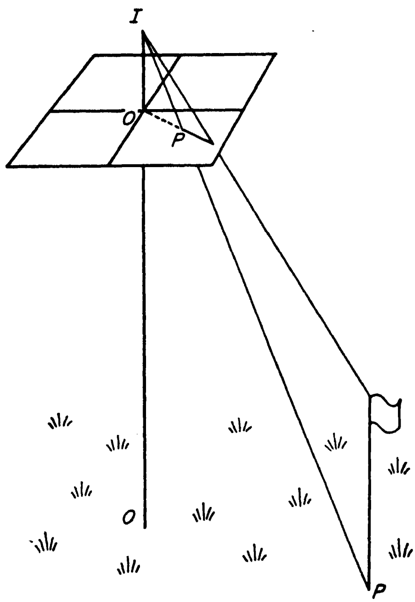


FIG. 16-2. Photographic Image of a Flagpole.

perpendicular to the picture plane, and therefore the image is a straight line which, if extended, passes through the principal point O. (2) In Fig. 16-5 are shown a vertical aerial view and the image

p_h of the top of a hill P_h. The vertical projection of P_h down to sea-level datum is at P_o, and the image of this point, if it could be seen in the photograph, would appear at the point p_o. Accordingly, the image of the vertical line P_hP_o is p_hp_o, and this line, if extended, passes through the principal point O. The length of the image d is called the *displacement* of the point p_h because of the elevation of the ground point above the datum plane. (3) In Fig. 16-3 are shown the images of two roads whose alinements on the ground are straight and which intersect at right angles as they pass over a hill. The principal point of the photograph is at O. The displacement of the points on the roadway, whose direction is radial from the principal point O, is along this same direction, and therefore there is no change in the alinement of the photographic image which remains a straight line. The points in the image of the other roadway are displaced radially from the principal point, and therefore the image of this roadway shows a convex curvature away from the point O.

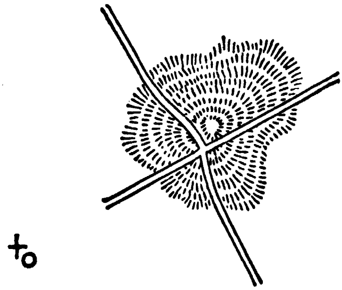


FIG. 16-3. Effect of Relief Displacement

16-4. The Scale of a Photograph It has been shown that the images of ground points are displaced where there are variations in the ground elevation. Hence, there is no uniform scale between the many points on such a photograph; therefore, in discussing the "scale of a photograph" in this article it is assumed that the ground is perfectly horizontal and the camera axis is truly vertical.

The scale of a photograph is the ratio of a given distance on the photograph to the corresponding distance on the ground. Where English units of measure are in use, this ratio is expressed in either of two forms, which may be designated as R, the representative fraction, or S, the map scale. Thus, in Fig. 16-4 for the sea-level elevation,

$$R = \frac{l}{L} \tag{16-1}$$

in which both l and L are expressed in the same unit, or the same ratio may be expressed as

$$S = \frac{L}{l} \tag{16-2}$$

in which L is expressed in feet and l is in inches. For example, if $l = 3$ in., and $L = 4500$ ft, then

$$R = \frac{0.25\ \text{(ft)}}{4500\ \text{(ft)}} = \frac{1}{18{,}000}; \text{ and } S = \frac{4500\ \text{(ft)}}{3\ \text{(in.)}} = 1500 \text{ ft per in.}$$

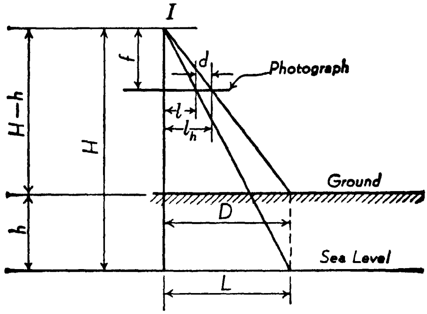


FIG. 16-4. Scale Relations in a Vertical Photograph.

The relationships between the scale of a photograph, the focal length f, and the height of lens H are also shown in Fig. 16-4. It is evident from the similar triangles that

$$\frac{l}{L} = \frac{f}{H}$$

and hence

$$R = \frac{f\ \text{(ft)}}{H\ \text{(ft)}} \tag{16-3}$$

also

$$S = \frac{L}{l} = \frac{H\ \text{(ft)}}{f\ \text{(in.)}} \tag{16-4}$$

Moreover, for the ground level at an elevation h above sea level, it is evident that

$$R_h = \frac{l_h}{L} = \frac{f}{(H - h)} \tag{16-5}$$

and

$$S_h = \frac{L}{l_h} = \frac{(H - h)}{f} \tag{16-6}$$

For example, if $f = 6$ in., $H = 9000$ ft, and $h = 500$ ft, then

$$R_h = \frac{0.5}{8500} = \frac{1}{17{,}000}; \text{ and } S_h = \frac{8500}{6} = 1417 \text{ ft per in.}$$

The latter scales are spoken of as the *representative fraction*, or the *scale*, of the photograph at the elevation h.

16-5. The Datum Plane It is evident in Fig. 16-4 that the ground distance D is represented by two different distances on the photograph, l and l_h corresponding to the two different elevations, sea level and h above sea level. Since the ground surface is usually characterized by slopes, hills, and valleys, it is obvious that the distance on a photograph that represents a given distance on the ground varies with the different elevations, and there can be no single scale that will apply to all points appearing in the photograph. Thus, to obtain the true ground distance between points of different elevations, it is necessary to refer them to a single plane of reference called the *datum plane*. For special conditions, any assumed elevation may be used as a datum plane; but since sea level is the universal datum for ground elevations it is also commonly used as the datum for aerial photographs. Therefore, unless otherwise specified in the following pages, H will represent the height of the lens above sea level, and h will represent the elevation of a ground point above sea level in all relations dealing with scale factors.

16-6. Number of Photographs Required Because of the overlap required in aerial photographs for mapping purposes, the net area covered by a single photograph will be that included within its full dimensions diminished by the overlap of adjacent prints.

The amount of overlap for two adjacent prints in the direction of the line of flight, called *forward lap*, is usually 60%. The distance between two principal points in a flight series is equal to the size of a print less the amount of forward lap. Thus, if the size of the print is 7 × 7 in., and the forward lap is 60%, then the distance between two adjacent principal points will be 7 in. − (7 × 0.60) = 2.8 in. Likewise, the distance between the principal points of photographs in adjacent flights is given by the size of the print perpendicular to the line of flight, less the amount of the side lap. Thus, if the print is 7 in. wide and the side lap is 25%, then the distance between two adjacent principal points will be 7 in. − (7 × 0.25) = 5.25 in. The

number of photographs required, therefore, will be the number required for one strip times the number of strips.

EXAMPLE: A flight mission is to be flown under the following conditions: The area is rectangular, 15 miles by 10 miles in size; the camera negatives are 7 in. square, and the focal length is 6 in. The scale of the photographs will be approximately 1500 ft per in. The forward lap is 60%, and the side lap is 25%. How many photographs will be required?

SOLUTION: The distance between the principal points of two photographs in the line of flight will be 7 in. $-$ $(7 \times 0.60) = 2.8$ in. $\times$ $1500 = 4200$ ft. The total length of one flight is $15 \times 5280 = 79{,}200$ ft. Hence the number of photographs required for one flight is $\frac{79{,}200}{4200} = 18.8$ or 19. The distance between two flights will be 7 in. $-$ $(7 \times 0.25) = 5.25 \times 1500 = 7875$ ft. The number of flights will then be $\frac{10 \times 5280}{7875} = 6.7$ or 7 flights. The number of photographs then is $7 \times 19 = 133$.

16-7. Image Displacement Caused by Ground Relief If conditions are as shown in Fig. 16-4, i.e., the photograph is truly horizontal and the ground is level, and if other sources of error are disregarded, then the photograph represents, at its proper scale, a true orthographic projection; hence it may be said to be a true map of the ground surface. Also, the photograph will have the same scale throughout the area contained within it. However, these conditions are never fully met in practice, and, since the photograph is a perspective view, any relief of the ground surface will be shown in perspective. Because of this condition, points in the photograph are said to be *displaced* from their true orthographic positions.

The displacement of an image caused by ground relief is shown in Fig. 16-5, where the image of a point P_h on a summit is shown at p_h, and the image of the vertical projection of this point to the datum plane P_o, is shown at p_o. The photographic distance $p_h p_o = d$ is the displacement of this image due to its elevation h above the datum.

If we let $op_h = l_h$, $op_o = l$, and $OP_o = OP_h = L$, the following equations may be written:

$$\frac{l}{L} = \frac{f}{H}; \text{ and } \frac{l_h}{L} = \frac{f}{H - h} \text{ also, } d = l_h - l = \frac{Lf}{H - h} - \frac{Lf}{H};$$

from which

$$d = \frac{l_h h}{H}$$

EXAMPLE: The distance from the principal point to an image on a photograph is 2.143 in., and the elevation of the object above the datum (sea level) is 850 ft. The height of lens above the datum is 7200 ft. Then $d = \dfrac{2.143 \times 850}{7200} = 0.253$ in.

Another example of the use of this relation is shown in Fig. 16-6 where the image of the top and the bottom of a tower, P_2 and P_1, are

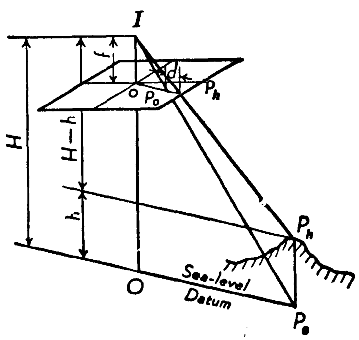


FIG. 16-5. Image Displacement in a Photograph.

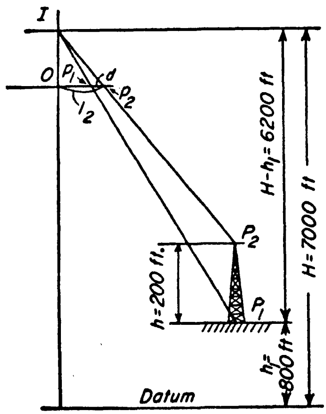


FIG. 16-6. The Scale of a Photograph.

shown on a photograph at p_1 and p_2. The distance to the point p_2 from the principal point O is indicated as l_2. The other values given are $H = 7000$ ft, $H - h_1 = 6200$ ft, $h = 200$ ft, and $l_2 = 2.120$ in. It is desired to find the displacement of the image of the top of the tower with respect to the image of the bottom, p_1. Following is the computation:

$$d = \frac{l_2 h}{H - h_1} = \frac{2.120 \times 200}{6200} = 0.068 \text{ in.}$$

16-8. To Find the Scale of a Photograph On any given photograph, if the images of two points appear whose ground elevations are equal and if the distance between them is known, then the scale of the photograph, for the known elevation h, is readily determined by the simple relation $S = L/l$, in which L is the known distance, in feet, on the ground, and l is the distance, in inches, on the photograph. The distance L may be measured on the ground, or if a published map is available it may be possible with sufficient accuracy to scale the distance from the map.

It will be seen, from Eqs. (16-1) to (16-6), that if the focal length of the lens is known, the height of lens H is readily computed, from which the scale of the photograph for the elevation h may be determined.

16-9. Computed Length of Line Between Points of Different Elevations From the method of the previous article and from the scale relations of Art. 16-4, it is possible to compute the correct horizontal length of a line between any two points of different elevation whose images appear on a photograph. For this problem it is necessary that the known data include the height of the lens H, the elevations of the two ends of the line, and the focal length f of the camera. The general procedure is as follows: (a) establish a system of coordinates for which the origin is the principal point, using the fiducial marks on the photograph; (b) scale the x- and y-coordinates for each end of the line; (c) compute the ground coordinates for X and Y for each of the image coordinates x and y; and (d) from the ground coordinate distances compute the length of the line.

Let A and B represent the ends of the line whose length is to be found, and a and b the images of these points. The elevations of these points are then h_a and h_b, respectively. From Eq. (16-6) the relation $S_a = \dfrac{H - h_a}{f}$ may be written, where S_a is the scale of the photograph for the point a, whose ground elevation is h_a above sea level. Likewise, $S_b = \dfrac{H - h_b}{f}$ expresses the scale for point b, whose ground elevation is h_b. Accordingly, if the x and y coordinates of the points a and b are scaled on the photograph, the corresponding coordinates of the points on the ground can be found as follows:

$$X_A = x_a \frac{H - h_a}{f}, \; Y_A = y_a \frac{H - h_a}{f}, \; X_B = x_b \frac{H - h_b}{f},$$

$$Y_B = y_b \frac{H - h_b}{f}$$

From these coordinates the true length of line AB may be computed, thus,

$$L = \sqrt{(X_A - X_B)^2 + (Y_A - Y_B)^2}$$

EXAMPLE: Given: $H = 12{,}325$ ft and $f = 8.25$ in.; also the scaled coordinates of two images a and b, and the elevations of these ground points A and B. Find the true horizontal distance of the line AB.

SOLUTION:

Point	h (ft)	$H - h$	$\frac{H - h}{f}$	Image	x (in.)	y (in.)	X (ft)	Y (ft)
A	720	11,605	1406.7	a	−2.174	−0.123	−3058	−173
B	915	11,410	1383.0	b	+1.424	−2.281	+1969	−3155

$$X_A - X_B = -5027 \qquad Y_A - Y_B = +2982$$

$$L = \sqrt{(-5027)^2 + (2982)^2} = 5845 \text{ ft}$$

16-10. Determination of the Height of Lens, *H* Conditions sometimes provide the known length of the line connecting points A and B having different known elevations, and it is required to find the height of lens, H. By a reversed procedure from that of the preceding article, and by the method of approximations, the true height of lens, H, may be found. The procedure is as follows: (a) scale the coordinates of images a and b on the photograph and compute the distance $d = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}$; (b) compute the mean elevation for points A and B; (c) from Eq. (16-6) compute the height of lens, H', which is an approximate value of H; (d) by the method of Art. 16-9 compute the length L' of line AB using the value of H' thus found. A better value, H'', can now be found by the relation $H''/H' = L/L'$. (e) This second value of H'' can now be used to compute the distance L'', which will be very close to the true distance L. Thus, the value H'' is verified as being the true height of

lens, H. If necessary steps (c), (d), and (e) can be repeated to obtain a more exact value for H.

EXAMPLE: Using the data of Art. 16-9, find the true height of lens, H.

1. The coordinates of the images a and b are scaled and found to be $x_a = -2.174$ in.; $x_b = 1.424$ in.; $y_a = -0.123$ in.; $y_b = -2.281$ in. Then $d = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2} = 4.196$ in.

2. The mean elevation of A and $B = 818$ ft.

3. $H' - h = \frac{L f}{l \times f} = \frac{5845}{4.196} \times 8.25 = 11{,}492$, from which $H' = 11{,}492 + 818 = 12{,}310$ ft.

4. $H = 12{,}310$ ft, $f = 8.25$ in.

Point	h (ft)	$H - h$	$\frac{H-h}{f}$	Image	x (in.)	y (in.)	X (ft)	Y (ft)
A	720	11,590	1404.8	a	−2.174	−0.123	−3054	−173
B	915	11,395	1381.2	b	+1.424	−2.281	+1967	−3151

$$X_A - X_B = -5021; \qquad Y_A - Y_B = +2978$$

$$L = \sqrt{(-5021)^2 + (2978)^2} = 5838$$

5. $$\frac{H''}{H'} = \frac{L}{L'} = \frac{5845}{5838} \times 12{,}310 = 12{,}325 \text{ ft.}$$

6. Using the value $H'' = 12{,}325$ ft and the given values of step 4, the computed value of $L' = 5845$. Since this value checks the value given for L in Art 16-9, it is proved that $H'' = 12{,}325$ ft is the true height of lens.

16-11. Control for Photogrammetric Maps It would be expensive to establish the positions of a sufficient number of points by ground control methods so that each photograph in a series could be accurately oriented in drawing the map; accordingly, graphical methods have been devised whereby the successive photographs can be oriented in the drafting room, with respect to each other and to ground features. The system of points thus fixed is termed *map control,* and a number of such systems have been devised, but the one most commonly used is known as the *radial-line* method.

16-12. The Radial-Line Method The radial-line method is the most accurate means of plotting a planimetric map from aerial photographs without the use of expensive instruments. It is based on three general principles which have been previously stated: (a) the displacements in a photograph due to ground relief are radial from the principal point; (b) images near the principal point are shown in their true orthographic positions, regardless of ground relief or tilt; and (c) the position of a point is correctly located on a map where three rays from three known points intersect.

16-13. Ground and Map Control The ground control should consist of the located positions of such objects as road intersections, lone trees, bridges, and fence corners, whose images can be accurately located on the photographs. The amount of control required will be determined by the scale of the map and the accuracy required. At least three points should be located which will show in the overlap area of the first two photographs of a flight series, somewhat as shown in Fig. 16-7. It is desirable that the three points P_L, P_C, and P_R shall be distributed evenly across the photograph transverse to the line of flight. If templets are to be used, it is desirable that the middle control point P_C shall be located far enough from the principal point O so that the slots and studs can be placed over both points simultaneously.

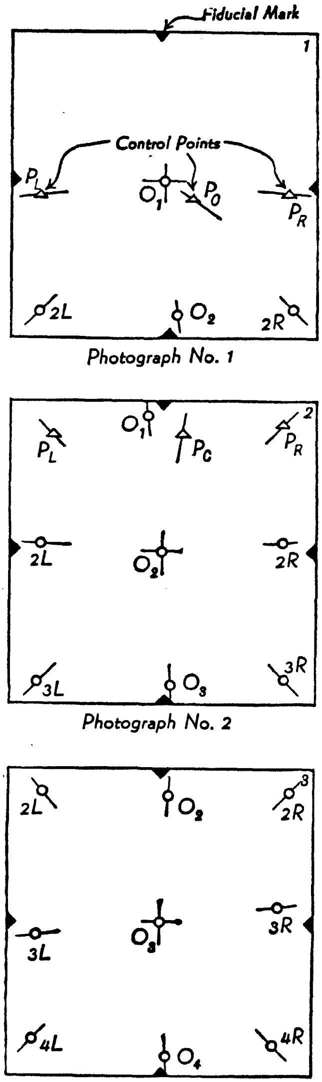


Fig. 16-7. Control Points from Overlapping Photographs.

It is advantageous to establish the ground control after the photographs

have been taken, because the character and distribution of the images will indicate where the ground points should best be located. However, if the control surveys have been run previously, they can be supplemented by a few ties and extensions to suitable objects which can be selected after the photographs have been taken.

16-14. The Base Map The base map can be drawn on heavy paper of high quality, on tracing cloth, or on cellulose acetate. A principal source of error in maps of some size is the change in the dimensions of the paper or tracing material due to changes in the humidity and temperature of the air. For this reason the acetate is a better tracing material than tracing cloth; if paper is used, and if high accuracy is desired, it is mounted on a metal base, for which aluminum is commonly used.

The coordinate grid which has been used in computing the position of the ground-control points is then plotted with great care on the base map, after which the positions of the control points are also plotted. The scale should approximate closely that of the photographs. Then, after the data in the photographs have been plotted to this scale, the map as a whole can be enlarged or reduced by means of a projector, by pantograph, or by photography.

16-15. Plotting the Line of Flight The principal point of an aerial photograph is readily found by means of two intersecting lines drawn between the opposite fiducial marks which appear on the margin of each photograph. For many uses of aerial photographs it is also necessary to locate on a given photograph the line of flight as determined by the plotted position of the principal points of the two adjacent pictures, one to the rear, and one forward in the direction of the line of flight. For this purpose it is therefore necessary to transfer the principal point of one picture to the next adjacent picture. This is done with the aid of a stereoscope. The principles of stereoscopic vision and fusion are discussed in Arts. 16-24 and 16-25, and should be read and practiced somewhat before any careful photographic measurements are made.

To obtain a stereoscopic view from two adjacent aerial photographs they must be oriented correctly with respect to each other, and the line of flight on the pictures must be parallel with the two lenses of the stereoscope. Then the distance between the two photo-

graphs must be adjusted until fusion occurs, when the relief in the landscape will be clearly visible.

To transfer the principal point of one photograph to the next one adjacent, it is, of course, necessary first to locate the principal point of each photograph as explained above. Then with the two pictures in their correct positions under the stereoscope, the principal point of one photograph will be seen directly and its image will be projected upon the other photograph. Then with a needle the position of the point can be transferred to the adjacent photograph.

If the principal point happens to be in a body of water or a smooth open field, with no definite objects near by, it will be difficult to transfer it by the method given above. For this condition two strips of acetate or other clear tracing material of suitable size should be provided. On each strip near its center a right-angle cross made by two fine lines intersecting each other should be ruled. The lines of the crosses should be approximately 2 in. long and make angles of about 45° with the line of flight. It may be assumed that the left-hand photograph contains the principal point which is to be transferred to the right-hand photograph. Then each cross is slid under the stereoscope and the left-hand one is made to coincide with the left-hand principal point. The other cross is then brought into fusion over the right-hand photograph. In this position the single fused point may appear to be buried below the ground surface, or to be floating above it. In this case, the right-hand cross is moved along the line of flight either way from or toward the other cross, until the fused single cross appears to rest upon the ground. Then the right-hand cross will be in its true position, and the principal point may be pricked through onto the photograph. If the image of the crosses should appear to be split or double, the right-hand cross must be moved up or down, perpendicular to the line of flight, until a single image is found.

16-16. Marking the Photographs The photographs are prepared for establishing the map control by marking on each one the radial lines and points which may be indicated by reference to Fig. 16-7, which shows the markings for the first three photographs of a flight series.

On photograph No. 1, the images of the ground-control points are identified and marked with needle points at P_L, P_C, and P_R, enclos-

ing each within small triangles drawn with a soft, colored pencil or with ink. Short lines radial from the principal point of the photograph are drawn through each control point. The same points are also marked on photograph No. 2. The principal point and the transferred principal points of the adjacent photographs are marked on each photograph, as shown at O_1, O_2, and O_3 of photograph No. 2.

On photograph No. 1 two additional points *2L* and *2R*, called *pass points*, are selected near the left and right edges of the photograph and approximately in line with the transferred principal point of the adjacent photograph.

On photograph No. 2 images P_L, O_1, and P_R appear near the upper edge; these and the control point P_C are then marked, and also the pass points *2L* and *2R* previously selected on photograph No. 1. Again two pass points *3L* and *3R* are selected near the lower left and right corners of photograph No. 2. Short lines radial from the principal point of photograph No. 2 are then drawn through each point. This completes the marking of this photograph.

In a similar manner each succeeding photograph is marked until other ground-control points are reached.

16-17. Plotting the Map Control After the photographs have been marked they are ready to be used in plotting the map control. For this purpose either tracing cloth or acetate may be used. A special quality of acetate can be procured which exhibits very small changes in its dimensions with changing atmospheric conditions. A matte surface is also provided which facilitates the drawing of fine pencil lines.

The plotting of the map control begins with the transfer of the plotted positions of the ground control from the base to the tracing sheet, by pricking through with a needle. Each point is then marked as shown at P_L, P_C, and P_R, Fig. 16-8. Photograph No. 1 is then slid under the tracing and adjusted until the radial lines through P_L, P_C, and P_R on the photograph, pass through the corresponding control points on the tracing. The photograph is then correctly oriented, and all radial lines upon it are traced upon the tracing, including the transferred principal point O_2.

Photograph No. 2 is then slid under the tracing and adjusted so that the lines previously drawn on the tracing pass through the corresponding points on the photograph, always keeping the lines along the line of flight, O_1O_2, in coincidence. When this orientation is com-

pleted, the rays through $3L$, O_3 and $3R$ on the photograph are traced on the tracing.

In this manner photograph No. 3 is now placed under the tracing as explained above, and its radial lines are traced.

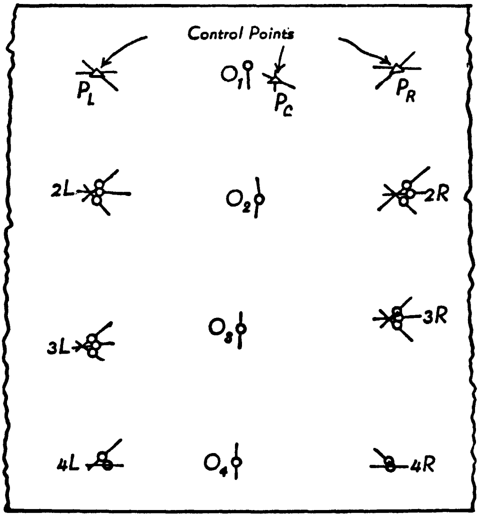


FIG. 16-8. Tracing Cloth or Acetate Sheet.

There will now appear three intersecting rays at pass points $2L$ and $2R$ on the tracing. If the three rays thus drawn intersect at a point, its position is correctly located on the tracing. Usually, however, this point of intersection will not appear to coincide with the corresponding point on the photograph, because the scale of the map is not the scale of the photograph, and because of other displacements due to ground relief and tilt. This condition is shown at pass point $2L$, Fig. 16-8. Here the images of this pass point, as traced from each of the three photographs, are shown by small circles, and the true position of this point on the control map is at the intersection of the three rays.

Also, because of tilt or errors in plotting, it will occasionally happen that the three rays for a given point will not intersect at a point, but will form a small triangle. In this case the position of the point is taken at the center of the triangle.

The procedure is continued until another ground-control point is reached. Here, if the map control is perfectly done, the image of the

control point, as located by the intersecting rays, will coincide with the plotted position of this point on the tracing. However, this will seldom happen, and the usual condition is shown in Fig. 16-9, where P is the plotted position of the ground-control point, and P' is the position of the same point as located by the intersecting rays of the map-control plotting. In this figure the principal points of only seven adjacent photographs, as located by the map-control plotting, are shown. The distance and direction of the line PP' may be regarded as the total error of plotting the traverse O_1 to O_7; hence, the adjusted positions of the points O_2 to O_7 are found by moving each plotted point along a line parallel to the line $P'P$, a distance proportional to the distance of that point from the initial fixed point O_1. Thus O'_7 is adjusted to O_7 a distance equal to $P'P$, and O'_6 is adjusted to O_6 a distance equal to five-sixths of $P'P$, and so on. Also, the positions of all other pass points will be adjusted accordingly.

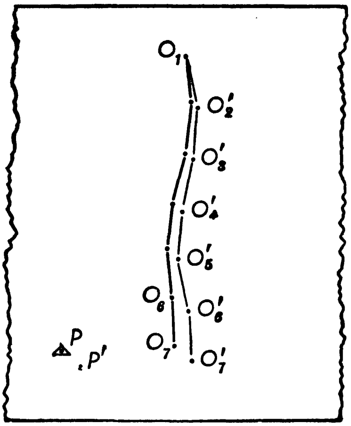


FIG. 16-9. Adjustment of Control Points.

16-18. Plain Templets If a considerable area is to be plotted by the radial-line method, the procedure can be facilitated by the use of templets, which are of two kinds, namely, *plain* and *slotted.*

The plain templets are of transparent material, preferably acetate sheets, 8 × 10 in. in size, one for each 7 × 9 in. photograph. Both the ground- and map-control points are selected on each photograph as for the radial-line method and marked by needle points, each with a small ink circle around it. Then the acetate sheet is placed over it, and the position of the principal point is pricked through onto the templet. Radial lines are now drawn on the templet from the principal point over each control point previously marked on the photograph.

A base map is prepared as described previously, using a scale which is averaged as nearly as possible for the photographs of the area to be mapped.

The first templet is then laid over the base map and adjusted to the ground control points, after which the second and third templets are also adjusted by the method previously explained; they are then fastened together by Scotch tape or by other suitable means. Each subsequent templet is added and fastened in position to the assembly of templets until the next set of ground-control points is reached. Now the ground-control points on the templets should coincide with the points on the base map; but because of the various sources of error, it will be only accidental when this coincidence is exact. The adjustment is then made by stretching or twisting the combined as-

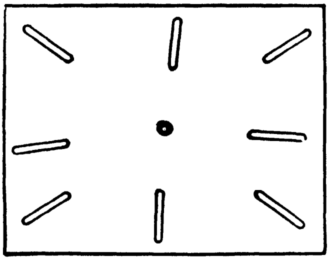


FIG. 16-10. Slotted Templet.

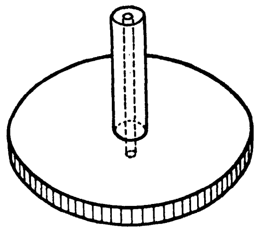


FIG. 16-11. Stud.

sembly of templets as a whole until coincidence of both ground- and map-control points has been established. The system of points which has been thus adjusted is transferred to the base map by pricking through the acetate with needles, or by laying a large transparent sheet over the templets and tracing the points upon it.

The detail of the photographs is then transferred to the base map as described in Art. 16-21.

16-19. Slotted Templets An improvement over the plain templet is the slotted templet. This method makes use of templets, studs, and a tool for punching holes and cutting slots in the templets. The latter are sheets of thin, firm cardboard about the same size as the photograph.

The pass points are marked as for a plain templet. Then each templet is taken to the punch, where a hole is punched at the principal point, and a slot is cut along the center line to each pass point.

The metal stud is a flat circular disc with a cylindrical post normal to its center, Fig. 16-11. There is a small hole drilled through the axis of the stud so that its position on the map can be marked by a needle pressed through it into the paper.

The ground-control points are plotted either on drawing paper or on thin metal (aluminum) sheets. A stud is then glued or soldered on each of the ground-control points on the map base.

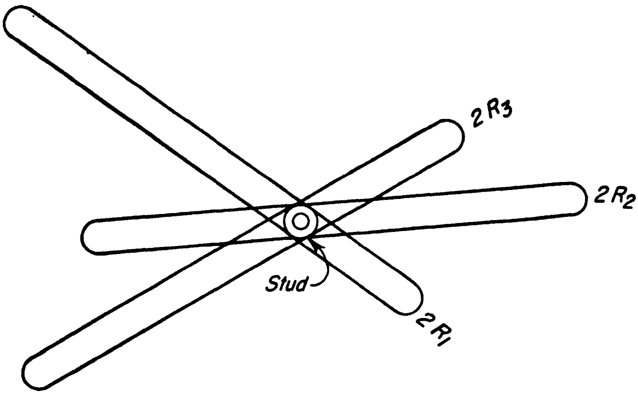


FIG. 16-12. Stud and Slot Assembly.

Templet No. 1 has slots cut for all pass points and has a round hole punched to fit each ground-control stud and the principal point. All succeeding templets are similarly prepared.

The assembly of the templets begins by placing templet No. 1 to fit the fixed ground-control studs on the base. Under each slot a stud is placed which is then movable along that slot. Each succeeding templet is placed so that each slot has its correct position relative to the assembly as a whole and so that each movable stud for each pass point is contained by the three slots intersecting at that point. Figure 16-12 shows the condition at the middle pass point on the right side of templet No. 2. The final assembly will then appear like Fig. 16-8, but instead of three intersecting rays to locate each pass point, there will be three slots held by the single stud.

When the assembly reaches the next ground control point, where a stud has been glued or soldered in place, the whole movable assembly is adjusted so that the last templet will fit its control stud. When this final adjustment has been effected, a needle pushed through the center hole of each stud will locate its pass point on the map. It is this final mechanical adjustment that gives this method its great advantage over other methods of plotting by the radial-line principle, for the templets are so interlocked from one end to the other end of the series that the final adjustment of the

last templet to its fixed control stud, simultaneously adjusts proportionately all pass points in the series.

If any photograph is badly affected by tilt, its templet will not fit the assembly but will be warped. Accordingly it will need to be retaken, and the tilt effect removed before it can be used. Also, any mistake resulting in a faulty slot position will likewise be detected when the templets are assembled. These mechanical means of detecting faults and mistakes constitute another important advantage of this method.

The tolerance between the studs and the slots is of such precision that a considerable number of templets can be assembled with great accuracy, which results either in a more accurate map or in the need for a less amount of ground control. Where manual drafting is employed, it is considered necessary to provide ground control at intervals of not to exceed eight photographs in a series; but with slotted templets the ground-control interval has been extended to 25 or more with excellent results.

After one flight series has been completed the next adjacent series is assembled, and the slots along the edges common to both series provide a means of tying the two flights together in one interlocked system of templets. When the entire system of templets has been assembled and adjusted, the position of each stud represents the correct position of each pass point, and its position is marked on the base map by pressing a needle through the hole in the stud, into the map beneath.

16-20. Spider Templets A further development of the templet method is that which, instead of a cardboard templet, makes use of thin, flat metal arms with slots cut in them, through which studs can be inserted as in the slotted templets. A bushing is provided for the stud which marks the principal point of a photograph. The outside of the bushing is threaded to receive a lock-nut. One end of each arm has a hole which fits the bushing. By this means an arm can be placed over a bushing and is then free to swing about it in a horizontal plane in any direction. When the desired number of arms have been thus placed they can be secured tightly by turning down a lock-nut over the bushing. Such a unit is called a *spider templet*.

Each templet is formed by inserting a needle at the principal point of a photograph and then the stud and bushing are placed over it. An arm is then placed over the stud and is swung in azimuth until

the center of the slot is precisely over one end of the radial line drawn on the photograph. This is done for each radial line, and in this position the group is secured by the lock-nut. Thus a templet is made for each photograph.

The templets are assembled as the slotted templets described previously, beginning with the fixed studs over the control points on the base map. The net is extended until the next control point is reached, when a complete and accurate adjustment of the whole system between control points is made. The position of each principal point and each pass point is then marked on the base by means of a needle pushed through each stud.

16-21. Transferring Photographic Detail When the map control has been plotted and adjusted, the details from each photograph are then transferred to the base map by one or another of various methods. Three commonly used methods are (1) by tracing, (2) by pantograph, and (3) by projection.

If the map is to be drawn to the same scale as the base map, the detail can be transferred by tracing directly from the prints. By this method, a photograph is placed under the tracing, adjusted to the map control, and the detail is traced upon the acetate sheet. In this work the photograph will be shifted from time to time as the draftsman works from one area to another, thus always keeping the photograph closely adjusted to control points in the immediate area that is being mapped. After the detail has been drawn on the acetate sheet, it can be transferred to the base map by pantograph or by tracing over a carbon sheet. The map may then be finished by the use of the usual colors and symbols.

The pantograph may be used not only to transfer the details from the photographs but to draw the map to a scale different from that of the base map.

The best method of transferring detail from a photograph to a map is by means of a properly designed projector. (See Art. 16-38.) Such an instrument projects the image of a photograph upon a plane map sheet on a drawing table, and provides for changes in scale and for the elimination of the effect of tilt in the photograph. With such an instrument, the image of a photograph is projected upon the base map which has the map control plotted upon it. This image is then adjusted accurately to the map scale by bringing the images of the control points in the photograph into coincidence with their plotted position on the base map. It has already been shown that the scale

of a photograph changes as between areas of marked difference in elevation within its borders. Consequently, for such a photograph it will be impossible to bring all images of control points into coincidence with their plotted positions by one focusing of the projector; in this case, the adjustment will be made over a small area for which the scale is practically constant, and then adjusted to another area. The detail is drawn directly upon the map sheet.

16-22. To Locate the Principal Points of Photographs on a Published Map It is often desirable to transfer images of points and objects from a photograph to a map. This can be done by a simple application of the principle of radial-line resections. A sheet of acetate or other tracing material is laid over the photograph, and the principal point is traced through. Also, radial lines from the principal point are drawn on the acetate over images of objects which also appear on the published map, such as road crossings, stream junctions, bridges, and fence corners. At least three radials having good intersection angles are desirable. Then the acetate is placed over the map and shifted until the radial lines pass over the corresponding points on the map. In this position the tracing is correctly oriented, and the principal point is pricked through onto the map.

16-23. To Transfer Images from a Photograph to a Map The position of any object whose image appears on a photograph can be transferred to a map by the use of intersecting radials, just as pass points are located when establishing map control. The images must appear on at least two, and preferably three, adjacent photographs. A tracing or templet for each photograph is made by drawing the principal point and radials as described in the preceding article, and also by drawing radial lines over each image which is to be transferred to the map. Then each templet is oriented on the map by the method of the previous article, and the principal point is pushed through onto the map. Also, when the templet is correctly oriented, the direction of the other radials is transferred to the map. When this procedure has been completed for two adjacent photographs, there will be two radial lines fixed on the map for each image to be transferred: one drawn from one principal point and the other drawn from the next adjacent principal point. The intersection of these two rays fixes the correct position of the desired point on the map. If a third intersection can be obtained from a third photograph, the position of the plotted point is verified.

16-24. Stereoscopic Vision It is a particular phenomenon of binocular vision that the observer is able to perceive spatial relations, i.e., the three dimensions of his field of view. This perception is due partly to the relative apparent sizes of near and far objects, and to the effects of light and shade, but an important condition is the fact that a given object is viewed simultaneously with two eyes which are separated in space; hence the two rays of vision converge at an angle upon the object viewed. The angle of convergence of the two rays of vision is called the *angle of parallax,* and its magnitude has an important effect upon the accuracy with which the observer can judge the true dimensions of a given object.

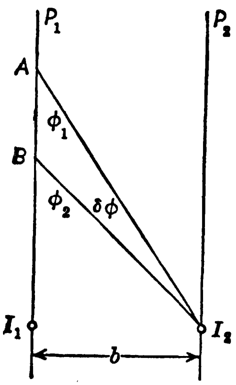


FIG. 16-13. Angles of Parallax.

Let I_1 and I_2 (Fig. 16-13) represent the positions of the two eyes of an observer, and let A and B represent two objects in the field of view. It is evident that the rays which converge upon A form the angle ϕ_1, and those upon B form the angle ϕ_2. Thus, as between two objects, that one will be judged to be nearer the observer for which the angle ϕ is the larger. It is further evident that $\phi_2 - \phi_1 = \delta\phi$, and this value is termed the *differential parallax.* This value is important since it provides a measure of the distance AB, in the line of vision, between two objects.

It is also evident that as $\delta\phi$ becomes small there is a limiting value below which the sense of stereoscopic vision is nil, and the observer is unable to judge, as between two objects, which is the nearer one. This limiting value of $\delta\phi$ for most observers is about 20″. If the distance between the observer's eyes is $2\frac{1}{2}$ in., then for an angle of 20″, the rays I_1P_1 and I_2P_2 meet at a distance of about 2100 ft; at that distance or beyond, the sense of stereoscopic vision becomes inoperative, and the relative distances to objects must be judged by their apparent size or by other factors.

However, the range and intensity of stereoscopic perception can be increased in two ways, either by apparently increasing the base between viewpoints or by magnifying the field of view by the use of lenses. Some binoculars use both of these principles, having prisms that apparently spread the base b of vision and lenses that magnify the field. If the base is thus apparently increased 2 times,

and if the lenses magnify the field 3 times, then the effect of stereoscopic perception is increased 6 times.

In aerial photogrammetry, stereoscopic vision is obtained by taking adjacent camera exposures from the moving airplane so that overlapping views of the ground are obtained. All of the principles of stereoscopic vision are then applicable within the overlap area of any pair of adjacent photographs.

16-25. Stereoscopic Fusion Two simple experiments will illustrate the phenomenon of stereoscopic fusion. Near the top edge of a sheet of paper draw four large dots about as shown in Fig. 16-14a,

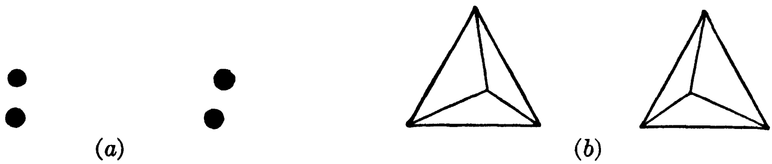


FIG. 16-14. Stereoscopic Fusion.

so that the horizontal distance between the bottom pair is slightly less than for the top pair. Hold a cardboard in the plane perpendicular to the sheet on which the dots are drawn, so that the right pair of dots is seen with the right eye, and the left pair is seen with the left eye. Then, holding the two sheets in this relation, focus the eyes on various objects at different distances behind the dots until the two pairs of dots fuse into one pair, one of the focused pairs being above the other. After a short time, as soon as the eyes have become adjusted to this fusion, one dot will appear to be nearer the observer than the other. This will be the bottom dot, if the distance between the bottom pair is the smaller.

A second drawing may be made as shown in Fig. 16-14b. Draw two equilateral triangles, whose centers are about 2 in. apart. For each triangle, substitute a different center, moved slightly toward the other triangle, and from each center thus chosen draw lines to the vertices of the triangles. Now observe this pair of figures in a manner similar to that described above, and when fusion of the two images has been obtained, the view will show a solid pyramid with the apex standing above the base.

In fact, with a little practice and without any screen, each pair of images shown in Fig. 16-14, a and b may be fused, and the stereo-

scopic perception will be very definite. Two marginal images will be apparent when observing each pair of images, but in the attention of the observer these can be disregarded.

16-26. Parallax in Aerial Stereoscopic Views The ideal conditions for obtaining aerial stereoscopic views of the ground surface are as follows: (a) two pictures are taken which overlap each other; (b) the elevation of the two camera positions, I_1 and I_2 is the same; (c) the camera axis is vertical, and therefore the picture planes (photographs) lie in the same horizontal plane. In Fig. 16-15, the

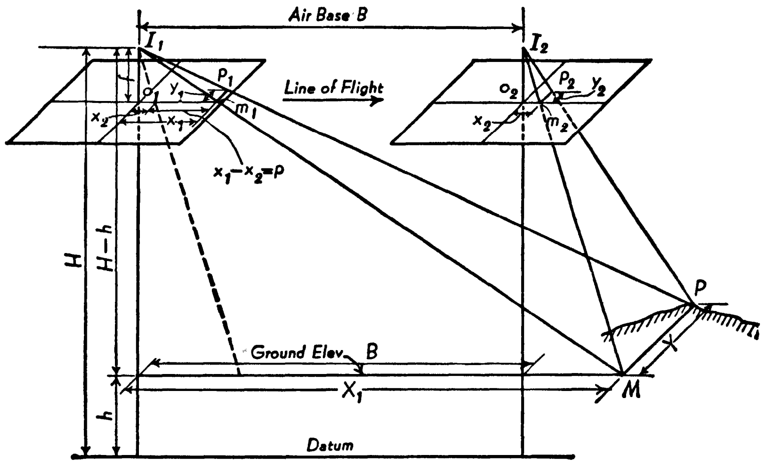


FIG. 16-15. Parallax in Aerial Stereoscopic Views.

two camera stations are shown at I_1 and I_2, being at the same height H above the datum. It is further assumed that the corresponding picture planes are truly horizontal and that the X-axis (principal line) of each is in the same vertical plane which contains the two camera stations. The line joining the two camera stations is called the *air base*.

The two images of an object P, which appear in both photographs, are p_1 for photograph I_1, and p_2 for photograph I_2. The x-coordinate of this object is x_1 for view I_1, and x_2 for view I_2, and the difference between the coordinates $x_1 - x_2 = p$ is called the ***absolute parallax*** of that image. It may be noted that for any stereoscopic pair of photographs for which the ideal conditions stated in the preceding

paragraphs exist, the parallaxes of the images for all points having the same elevation will be equal.

It should be noted that if point P were located between the two ground nadirs, i.e., to the left of camera station I_2, then the direction of the coordinate x_2 would be to the left of o_2 in the right-hand photograph. The sign of this coordinate would be negative and, therefore, in the equation $p = (x_1 - x_2)$ it would be added.

An important observation should now be made; there is no parallax perpendicular to the x-axis, i.e., there is no y-parallax in a stereoscopic pair of photographs.

16-27. The Space-Coordinate Equations for Aerial Stereoscopic Views In Fig. 16-16 the relations are shown whereby the

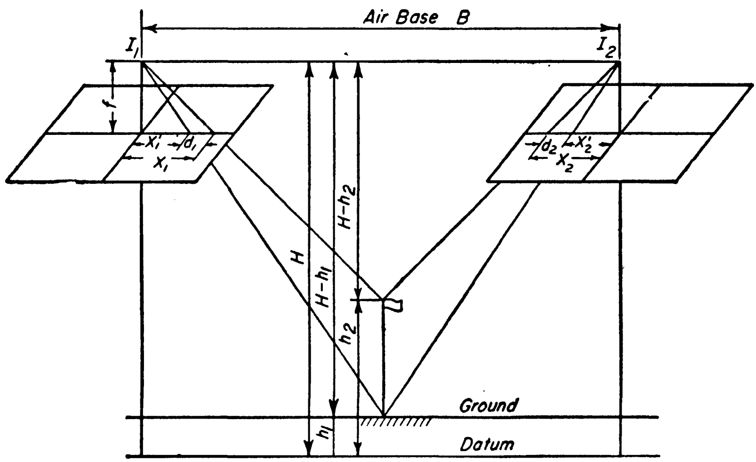


FIG. 16-16. Differences in Elevation by Stereoscopic Parallaxes

three space coordinates X, Y, and $(H - h)$ of any object with respect to the left-hand camera station are obtained from a stereoscopic pair of aerial views. From the similar triangles shown, the following equations can be written:

$$\frac{X_1}{x_1} = \frac{H-h}{f}, \text{ but } \frac{H-h}{f} = \frac{B}{x_1 - x_2} \text{ and since } x_1 - x_2 = p,$$

$$X_1 = \frac{B}{p} x_1 \qquad (16\text{-}7)$$

$$\frac{Y_1}{y_1} = \frac{H - h}{f}, \text{ but } \frac{H - h}{f} = \frac{B}{x_1 - x_2}; \text{ hence,}$$

$$Y_1 = \frac{B}{p} y_1 \tag{16-8}$$

Also,

$$\frac{H - h}{f} = \frac{B}{p}, \text{ or } (H - h) = \frac{Bf}{p} \tag{16-9}$$

These equations are of fundamental importance; they provide, for the ideal conditions stated previously, the means of determining the three space coordinates, and hence the true relative positions of all ground points within the overlap area of any adjacent pairs of photographs.

16-28. Difference in Elevation by Stereoscopic Parallaxes In Fig. 16-17 are shown the parallaxes and displacements which result

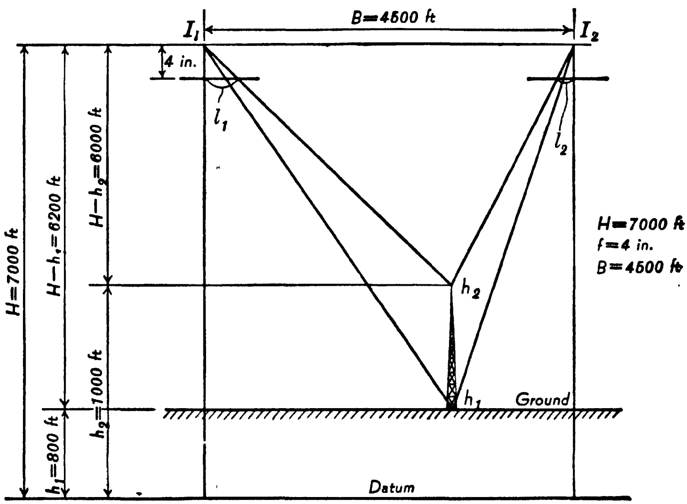


FIG. 16-17. Parallax Computations.

from the perspective rays drawn to the top and the bottom of a flagpole from each of two camera positions, I_1 and I_2. In this drawing the flagpole is assumed to be in the plane of the line of flight. If the pole were at another place not on the line of flight, anywhere in the stereoscopic view, the separate parallaxes x_1, x'_1, etc., would have

different values, but the parallaxes p_1 and p_2 would be the same. In this plane of the line of flight, the parallax p_2 of the top of the pole is $x_1 - x_2$, and the parallax p_1 of the bottom of the pole is $x_1' - x_2'$. The difference in these parallaxes is, therefore, $\Delta p = p_2 - p_1 = (x_1 - x_2) - (x_1' - x_2')$. In the figure the value of Δp is shown as $(d_1 + d_2)$. Thus, it is evident that the parallax difference Δp is equal to the displacements $(d_1 + d_2)$ of the top of the flagpole with respect to the bottom measured in the direction of the line of flight.

In Eq. (16-9) it is obvious that the elevation of any object above the datum can be expressed in terms of the parallax value for that point, thus,

$$h = H - \frac{Bf}{p} \tag{16-10}$$

Therefore, the elevations of the bottom and top of the flagpole may be written

$$h_1 = H - \frac{Bf}{p_1} \quad \text{and} \quad h_2 = H - \frac{Bf}{p_2}$$

and the difference in elevations

$$\Delta h = h_2 - h_1 = \left(H - \frac{Bf}{p_2}\right) - \left(H - \frac{Bf}{p_1}\right)$$

$$\text{or } \Delta h = \frac{(p_2 - p_1)Bf}{p_1 p_2} \tag{16-11}$$

16-29. Parallax Computations The relations expressed in Eqs. (16-9), (16-10), and (16-11) are shown in the computations which follow (see Fig. 16-17).

Given: $B = 4500$ ft; $H = 7000$ ft; $f = 4$ in.; $h_1 = 800$ ft; $h_2 = 1000$ ft; $l_1 = 2.000$ in.; $l_2 = 1.000$ in. Required: p_1, p_2, d_1, d_2, and Δh.

$$p_1 = \frac{Bf}{H - h_1} = \frac{4500 \times 4}{6200} = 2.903 \text{ in.} = \text{absolute parallax for } h_1$$

$$p_2 = \frac{Bf}{H - h_2} = \frac{4500 \times 4}{6000} = 3.000 \text{ in.} = \text{absolute parallax for } h_2$$

$$d_1 = \frac{l_1 h}{H - h_1} = \frac{2.000 \times 200}{6200} = 0.0645 \text{ in.}$$

$$d_2 = \frac{l_2 h}{H - h_1} = \frac{1.00 \times 200}{6200} = 0.0323 \text{ in.}$$

$$\left.\begin{aligned} \Delta p &= 3.000 - 2.9032 = 0.0968 \\ d_1 + d_2 &= 0.0645 + 0.0323 = 0.0968 \end{aligned}\right\} \text{check}$$

$$\Delta h = \frac{(p_2 - p_1)Bf}{p_1 p_2} = \frac{0.0968 \times 4500 \times 4}{3.000 \times 2.9032} = 200.0 \text{ ft}$$

16-30. Effects of Changes in Elevation h and Parallax p The basic equation $H - h = Bf/p$ expresses the relation between the height of lens $(H - h)$ above any object and the absolute parallax p; and Eq. (16-11) above provides the means of finding the difference in elevation between any two objects whose images appear in a stereoscopic pair of photographs. By this method, however, it is necessary to measure the two absolute parallaxes p_1 and p_2 separately, and then to compute the difference in elevation, Δh. If only a few computations are desired, this method is quite satisfactory; but in mapping, where many computations are necessary, it is more expedient to make use of the difference between the two absolute parallaxes, i.e., $\Delta p = p_2 - p_1$. This method makes use of instruments which measure the value of Δp directly by means of micrometer scales, and the fusion of two dots in the stereoscopic view into a so-called *floating mark* (Art. 16-41). It is not within the scope of this chapter to describe these instruments in detail, but the theory will be given so that, with a few directions, any one of the instruments may be used. Reference is made to the *wedge*, the *parallax bar* and the *stereocomparagraph.*

The value of Δh may be derived as follows: from Eq. (16-9), Art. 16-27, $h = H - (Bf/p)$; then, by differentiation,

$$dh = \frac{Bf}{p^2}\,dp \tag{16-12}$$

Again from Eq. (16-9), $p = Bf/(H - h)$, and if this value of p is substituted in Eq. 16-12, then

$$dh = \frac{H - h}{p}\,dp. \tag{16-13}$$

In this equation $(H - h)$ represents the height of lens above a given ground point, and p is the absolute parallax for the images of the same point in the stereoscopic pair of photographs. This equation states the rate of change in the computed elevation of any ground point corresponding to a small change in the absolute parallax of the point.

If it is assumed that this rate of change remains constant between the two points h_1 and h_2, then the difference in elevation between these points would be computed as follows:

$$\Delta h = \frac{H - h_1}{p_1} \Delta p. \tag{16-14}$$

in which p_1 is the absolute parallax of the low point h_1, and Δp is the difference in the parallaxes $p_2 - p_1$. But it is known that the relation between dh and dp is not a constant ratio, and therefore when applied to finite values, Eq. (16-13) is an approximate one. This can be proven by the example of the previous article.

Given: $H - h_1 = 6200$ ft, $p_1 = 2.903$ in., and $\Delta p = 0.0968$ in. Then $\Delta h = \dfrac{6200 \times 0.0968}{2.903} = 206$ ft. The true difference in elevation $h_2 - h_1 = 200$ ft and, accordingly, the error caused by using Eq. (16-13) is 6 ft. For most conditions where the instruments mentioned may properly be used, the error caused by the approximation is not important and may be neglected.

16-31. Other Approximations For the usual practical application of Eq. (16-14), Art. 16-30, two additional approximations are permitted which greatly expedite the work. (1) The approximate quantity H' is assumed to be the average height of the plane over the average elevation of the terrain in the stereoscopic view; and (2) the quantity p_1 is replaced by the quantity b, which is the mean value of the two map air bases; i.e., the mean of the photograph distances between the conjugate pairs of principal points. This latter approximation assumes that the average elevation of the nadir points below the two positions of the plane will closely approximate the average elevation of the terrain. Equation (16-14) then becomes

$$\Delta h = \frac{H'}{b} \Delta p \tag{16-15}$$

EXAMPLE: The average height of the plane over the terrain in a stereoscopic view is estimated to be 5900 ft. The average of the two map air bases is 3.14 in. The measured difference in parallax between two points, h_1 and h_2, is 0.075 in. Then the computed difference in elevation is $\Delta h = \dfrac{5900}{3.14} \times 0.075 = 141$ ft.

16-32. The Parallax-Table Method From Equation 16-9, $p = \frac{Bf}{(H-h)}$, in which it may be noted that $\frac{f}{(H-h)}$ is the inverse ratio of the scale of the photograph at the elevation h above the datum. Also, since B is the air base for the stereoscopic pair of photographs, it is evident that the absolute parallax p is equal to the air base B divided by the scale of the photograph at the elevation h, and this value is also equal to b, the distance on the photograph corresponding to the air base distance B. Therefore, $p = b$ for a given elevation h above the datum. These relations are shown in Fig. 16-18,

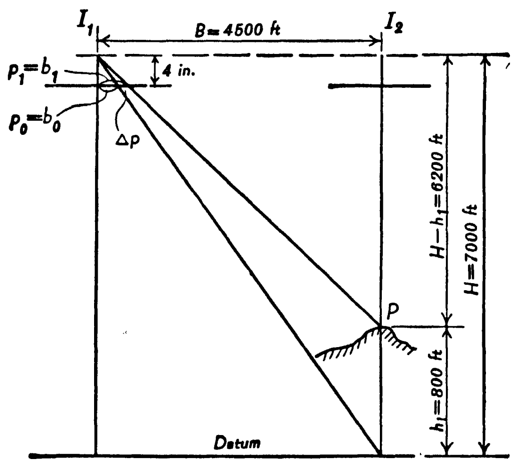


FIG. 16-18. Relations Between Parallaxes and Elevations.

where the ground object P is vertically below I_2. For this condition, $p_0 = b_0$, and $p_1 = b_1$, where b_0 and b_1 are the photograph distances of the air base B at the elevation of the datum and of h_1, respectively.

It is now evident that Δp is the change in absolute parallax $(p_1 - p_0)$ corresponding to the change in elevation from the datum to h_1. It is also evident that Δp may be regarded as the displacement of the image of P on the photograph, given by the equation $\Delta p = \frac{b_1 h_1}{H}$, in which b_1 is the absolute parallax of the point P at the elevation h_1

above the datum. It can readily be shown that this relation will be true for any position and for any elevation of the point P.

It may be noted here that the scales on most instruments used for measuring parallaxes are divided in the metric system. Accordingly it is often necessary to make use of the transformations, 1 mm = 0.03937 in., and 1 in. = 25.400 mm.

By means of the equation $\Delta p = \frac{b_1 h_1}{H}$, a table of total parallaxes (Table XV), designated as $\Sigma\Delta p$ has been computed for the following conditions: $H = 25{,}000$ ft; the absolute parallax $b = 100$ mm at the datum elevation; and the increments in h are 20 ft for the interval $(H - h) = 25{,}000$ ft to $(H - h) = 10{,}000$ ft. Then the increments are 10 ft for the interval from $(H - h) = 10{,}000$ to $(H - h) = 5000$ ft. The computation is as follows:

$$\Sigma\Delta p = \frac{bh}{H - h} \tag{16-16}$$

in which b is the absolute parallax at the datum elevation (i.e., $b = 100$ mm), h is the elevation above the datum, and $(H - h)$ is the height of the lens above the point for which $\Sigma\Delta p$ is being computed.

The values in the table may be verified by use of Eq. (16-16) as follows: Let $h = 7000$ ft; then $H - h = 18{,}000$ ft and

$$\Sigma\Delta p = \frac{bh}{H - h} = \frac{100 \times 7000}{18{,}000} = 38.889 \text{ mm}$$

The values of $\Sigma\Delta p$ in the table may be adapted to other conditions if they are multiplied by a constant K, such that

$$K = \frac{b(\text{photo}) \times H(\text{photo})}{100 \times 25{,}000}$$

where the values b and H are those for the given photograph corresponding to 100 mm and 25,000 ft in the table. Thus, for the conditions shown in Fig. 16-19 the value of $\Sigma\Delta p$ for each elevation h_1 and h_2 above the photograph datum will be found by taking the value in the table corresponding to $\Sigma\Delta p$ for each height of lens $(H - h_1)$ and $(H - h_2)$ and multiplying these by the value of

$$K = \frac{65.314 \text{ mm} \times 7000 \text{ ft}}{100 \text{ mm} \times 25{,}000 \text{ ft}} = 0.18288$$

where 65.314 mm $= b_1$, at the photograph datum.

Hence

$$\Sigma\Delta p_1 = 303.226 \times 0.18288 = 55.454 \text{ mm}$$

and

$$\Sigma\Delta p_2 = 316.667 \times 0.18288 = 57.912 \text{ mm}$$

The difference between these two values may be designated as Δp and is equal to 2.458 mm = 0.097 in.

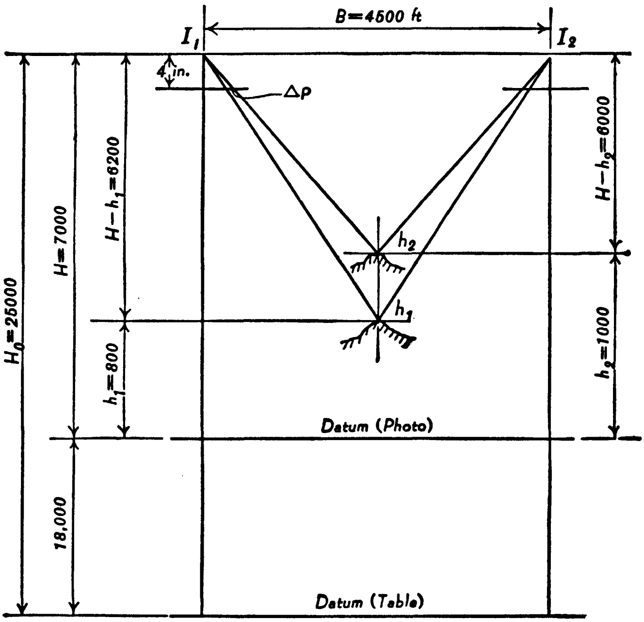


FIG. 16-19. Parallax-Table Method.

The value of Δp may also be computed as a displacement of the image h_2 with respect to h_1. The computation would be $\Delta p = \dfrac{b_2 h}{(H - h_1)}$, where b_2 is the absolute parallax or photograph distance for the air base B, at the elevation of h_2; h is the difference in elevation between h_1 and h_2; and $H - h_1$ is the height of lens above h_1. For the given data, $\Delta p = \dfrac{3.000 \times 200}{6200} = 0.097$ in.

It is obvious that the same value of Δp will be found if this computation is reversed and the displacement of h_1 is computed with respect to h_2. For this condition the computation is

$$\Delta p = \frac{b_1 h}{H - h_2} = \frac{2.903 \times 200}{6000} = 0.097 \text{ in.}$$

16-33. Spot Elevations A common problem is to find the elevation of a given object whose image appears in a pair of photographs, together with the image of another object or benchmark whose elevation is known. This may be done by the methods of Art. 16-28 and by the use of the parallax tables. Referring to Fig. 16-19 and Table XV, the computation may be arranged as shown. The given data are: $B = 4500$ ft, $H = 7000$ ft above the datum for the photographs, $h_1 = 800$ ft above the datum, and $b_1 = 2.5714$ in. $= 65.314$ mm at the photograph datum. It is required to find the elevation of h_2.

$H = 7000$ ft $\quad b = 2.5714$ in. $= 65.314$ mm

Point	Elevation	$H - h$	$\Sigma\Delta p$ mm (Table)	Micrometer
h_1	800.0	6200.0	303.226	0.000
h_2	1000.0	6000.0	316.698	2.464

The results are tabulated as shown in the table, and the procedure is as follows:

1. The values for h_1 are recorded as Elev. $= 800$, $(H - h) = 6200$, and, from Table XV, $\Sigma\Delta p = 303.226$.

2. The value of $\Sigma\Delta p$ for the photograph is obtained by multiplying the value in the table by $K = \dfrac{bH}{(b_0 \times H_0)}$, in which b is the absolute parallax at elevation of the photograph datum: H is the height of lens above this datum: b_0 is the absolute parallax at the parallax table datum; and H_0 is the height of the lens above this datum. Accordingly,

$$K = \frac{(2.5714 \text{ in.} \times 25.4 \text{ mm})(7000 \text{ ft})}{100 \times 25{,}000} = 0.18288$$

Then,

$$303.226 \times 0.18288 = 55.454 \text{ mm}$$

3. It is assumed that the micrometer scale reads 0.000 mm when the floating mark is on the ground at h_1. Then the mark is moved until it is on the ground at h_2, when the micrometer scale reads $\Delta p = 2.4638$ mm (theoretically), and is recorded.

4. The value of 2.464 mm is then multiplied by the reciprocal of K to yield the corresponding value of Δp in units of the parallax table. Thus, Δp (table) $= 2.464 \times 1/0.18288 = 13.472$ mm. This value is added to $\Sigma\Delta p = 303.226$ and recorded as 316.698, which is the value of $\Sigma\Delta p$ in the table for the desired elevation h_2.

5. By interpolation, this value of $(H - h)$ in Table XV corresponding to $\Delta p = 316.698$, is found to be 5999.6 ft; accordingly, $h_2 =$ 1000.4 ft. These values, for all practical purposes, are 6000.0 ft and 1000.0 ft, respectively, and are so recorded.

Thus, the value of point h_2 has been determined by means of the parallax table, and the measured difference in parallax, $\Delta p = 2.464$ in.

16-34. Mosaics A *mosaic* is an assembly of photographs of a given area, matched as nearly as may be and pasted upon a background to form a photographic representation of an area.

For some purposes where the character of the physical features in the terrain is more important than dimensions scaled from a map, a mosaic is more useful than a map.

Because of the large displacements of points near the edges of photographs, best results will be obtained where a large amount of overlap is maintained. Then the edge portions can be trimmed away and only the central areas, comparatively free from displacements, are used. Also, for the same reason, single-lens pictures usually yield a better mosaic than multiple-lens pictures.

Mosaics are classified as either *controlled* or *uncontrolled*, depending on whether or not some kind of control is used in building the mosaic.

An uncontrolled mosaic is suitable where only a limited area is to be portrayed. No ground-control points or other maps are used in the assembly of the prints. A central photograph is chosen, trimmed, and pasted to the base. The other pictures are then matched as closely as possible and pasted in place, until the mosaic is completed.

When some adherence to scalable dimensions is desired, the positions of the prints may be controlled either by ground measurements or by a reliable existing map. Such control must be plotted at the average scale of all the photographs as nearly as can be determined. A better procedure is to rephotograph the pictures and bring them all to the same scale.

Furthermore, where a general slope affects a considerable area, as

along the side of a valley, it has been found practicable by rectifying photography to obtain a photograph which represents the sloping area as though it were horizontal. Dimensions scaled from such photographs are practically free from displacements caused by relief.

After photographs have been thus prepared, they may be assembled by either of the methods explained in previous articles.

It must be added that, because of irregularities in the features of the terrain, no mosaic can be regarded as suitable as an accurate planimetric, or topographic, map for purposes of engineering designs involving linear dimensions.

16-35. Flying Because of the effects of tilt and variations in the height of the lens on the resulting photographs, it is important that the pilot keep the airplane on an even keel and at a constant elevation while exposures are being made. The use of gyroscopes and spirit-level bubbles on the cameras has not proved to be satisfactory, so reliance must be placed on the services of well-trained pilots and stable planes capable of operating at high altitudes.

It is also important that each flight shall be straight so that the overlap between adjacent strips shall be uniform and no gaps occur. This has been a difficult condition to meet, especially where no accurate existing maps of the area are available. However, the Solar Navigator has proved to be an excellent aid. It utilizes the sun's rays to enable the pilot to fly any predetermined course and has demonstrated that a course as long as 50 miles can be flown within a deviation of $\frac{1}{4}°$ from a given bearing.

16-36. Cameras Many factors must be considered in the design of a camera suitable for photogrammetric mapping, and underlying all of these is the resulting cost of the photography and the map. Since the cost is largely determined by the number of photographs required, every possible means is used to increase the ground area which can be included within a single exposure, the print of which will be suitable for drawing the map. The principal factors which must be considered are (a) the focal length of the lens, (b) the definition, and (c) the distortion in the photographs.

Scale Factors. The scale of a photograph is determined by the height (H) of the camera and the focal length (f) of the lens; also, the area represented by a photograph of given size will be proportional to H and inversely proportional to f. Theoretically, then, to

reduce the cost of the survey, f should be reduced and H should be increased. But if f is much reduced, the angle of the field of view becomes so wide that distortions near the edge of the picture are too great. Also, if H becomes too great the images are too small to be defined or recognized. Thus, there are limitations with regard to each factor, and the best design is that which keeps within proper limits with regard to each factor, and which yields satisfactory results at a minimum cost.

Maps are drawn at widely different scales, but, disregarding military surveys, the height at which an airplane can operate satisfactorily is limited between perhaps 5000 and 20,000 ft; hence a considerable range of focal lengths is required from about 4 in. to 20 in.

For topographic mapping it is important that ground parallaxes shall be as large as possible, and these will be obtained best with short-focal-length lenses, whereas longer focal lengths are more suitable for planimetric maps and are sometimes necessary for maps at large scales. Values commonly used are $5\frac{1}{4}$ in., and $8\frac{1}{4}$ in., respectively.

Definition Factors. Since airplanes are moving rapidly when exposures are made, it is essential that the shutter speeds shall be fast enough to prevent blur. These will depend also upon the height of the airplane, and they vary from 1/25 to 1/500 sec.

The scale of photographs is frequently increased by enlargement from contact prints, but the amount of increase is limited by the size of the emulsion grains. The finer the grain, the better is the definition; but since the larger grains produce a faster film, a compromise must be made which will yield satisfactory definition together with the necessary shutter speeds.

The resolving power of a lens has a direct effect on the definition obtained. This quality in a lens is that which renders apparent the separateness of very small images. Thus, a good lens will show as two separate images what a poor lens will show as a single, merged, indefinite image. The resolving power of a lens for an aerial camera should be definitely specified and tested.

Distortion Factors. There are many distortion factors which affect photographic images, some of which are inherent in the film and film base rather than in the camera.

Ordinary film base will shrink during the developing and drying

process, but low-shrink materials are now available which have reduced this source of distortion.

It is important that the film shall be perfectly flat and accurately in the focal plane of the lens at the moment of exposure. This is accomplished by means of a true surface platen against which the film is pressed, either by a glass plate or by air pressure caused by a vacuum acting through holes in the platen. Obviously, the faulty action of either of these mechanisms will cause distortions in the negatives.

The fiducial marks on the frame of the plate-holder, which fix the position of the principal point on each photograph, must be precisely set and maintained.

A principal source of distortion, however, is the lens itself, and for photogrammetric uses it should be calibrated at the U.S. Bureau of Standards to determine the magnitude of the distortions present, in all zones of the field of view.

16-37. Single-Lens Cameras The Abrams and the Fairchild companies have cartographic precision cameras designed especially for mapping use.

The Abrams camera has a frame made of invar metal to reduce the effects of excessive changes in temperature. It carries film for about 650 negatives $9\frac{1}{2} \times 9$ in. It has provisions for the use of 5, 6, 8, or 10 in. lenses interchangeably.

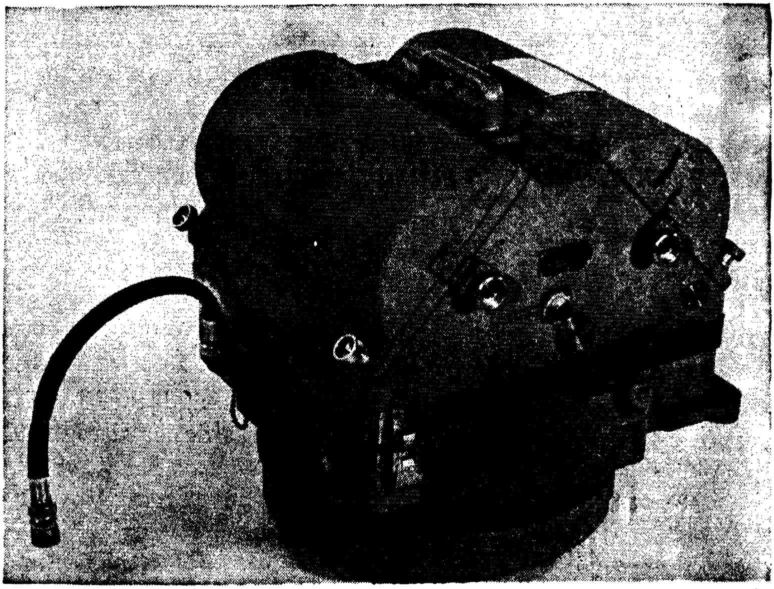
The Fairchild T-11 camera, Fig. 16-20, is equipped with a 6-in. Planigan lens, and Rapidyne shutter speeds from 1/10 to 1/500 sec. The magazine carries film for about 450 exposures 9×9 in.

These cameras are fully automatic in operation. The accessories include an intervalometer which controls and regulates the interval between exposures, and a viewfinder which assures the proper orientation of the camera with respect to the line of flight.

Upon request, the manufacturers will supply for each of these cameras a certificate from the U.S. Bureau of Standards giving the calibration of the lens and other information to insure the requisite precision in the photographs.

16-38. Photogrammetric Mapping. Projectors By the principles of perspective and stereoscopic vision explained in this chapter it is possible to construct a topographic map from aerial views, with

the aid of a few ground-control measurements, and with simple drawing room equipment. A map so drawn, however, must necessarily be rather crude and inaccurate. It is also possible by precise instrumental measurements of the coordinates of points which appear in



Courtesy of The Fairchild Camera and Instrument Corporation

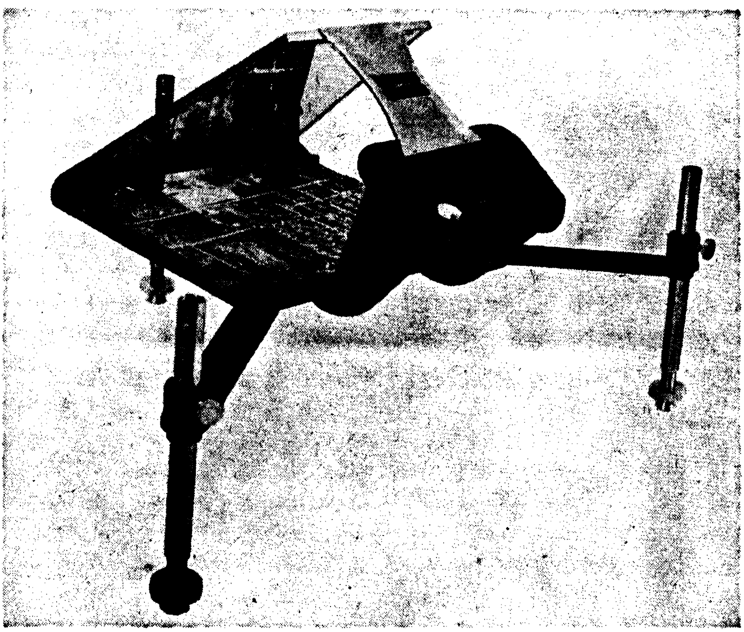
FIG. 16-20. The T-11 Fairchild Cartographic Camera.

photographs, together with suitable field measurements, to compute by analytical methods the three space coordinates of the objects whose images appear in the photographs. From these data, a map can be drawn by the usual drawing room methods.

However, because of the inadequacy of the first method and the computations required by the second method, more practical means of constructing maps from photographs have been sought by the use of instruments which achieve the desired results by mechanical and graphical means. Many such instruments are now available whose capabilities range from crude to highly precise results, and whose designs range from the comparatively simple to the exceedingly complex and intricate. The purpose of the following descriptions is to indicate, in general terms, the character of the principal instru-

ments and methods which are being used in photogrammetric mapping in the United States.

Various types of projectors have been designed to facilitate the transfer of planimetric features, such as roads and streams, from the photograph to the map. These instruments use the principle of the camera lucida, which by an arrangement of prisms and mirrors projects the image of a photograph upon a map where the features shown in the photograph can be traced directly upon the map. Some special designs are adapted to oblique views. Figure 16-21 illustrates



Courtesy of Abrams Aerial Survey Corporation

FIG. 16-21. The Sketchmaster.

the "sketchmaster" of the Abrams Aerial Survey Corporation.

Projectors are manufactured under such names as "Sketchmaster," "Rectoplanograph," "Planimetric Plotter," and "Multiscope."

16-39. To Orient an Instrument for Measuring Parallaxes All measurements of parallax in the overlap area of a pair of photographs must be made in a direction parallel with the line of flight.

This direction is fixed by the line connecting the principal point of one picture with the transferred image of the principal point of the adjacent picture. The following procedure applies particularly to the parallax bar, and the stereocomparagraph. The procedure is as follows:

Place the instrument over a plain sheet of paper and set the micrometer scale at about its mid-point. The y-adjustment scale should also be set at about its mid-point. Adjust the right hand index by the micrometer screw until the images of the two dots at the surface of the paper fuse and remain a single dot. Measure the distance between the two dots, d. Adjust the instrument over the left photograph with the straight edge along the line of flight. Slide the right photograph under the instrument so that the line of flight coincides with the straight edge and so that the distance between the principal point of the left photograph and its transferred image on the right photograph is equal to the distance d.

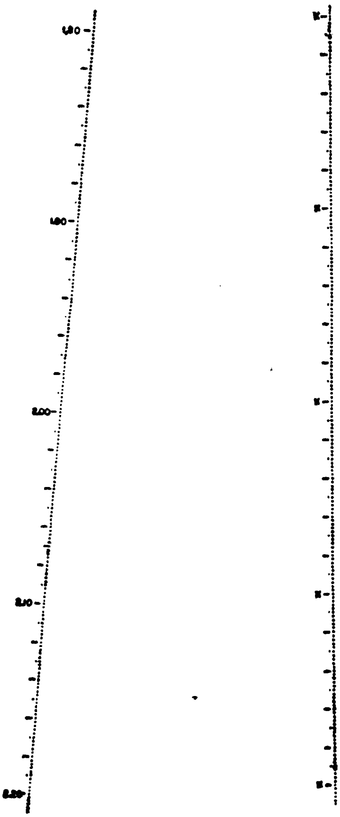


FIG. 16-22. The Parallax Wedge.

16-40. The Parallax Wedge

The parallax wedge is a simple device for measuring, with some precision, parallaxes in a stereoscopic pair of photographs (Fig. 16-22). It consists essentially of two fine lines printed on a transparency and converging at a small angle. The horizontal distance between the lines both at the top and at the bottom is precisely fixed. Each line is marked with an accurately divided scale which reads, at any given division, the horizontal distance between the lines. The dimensions may be marked by ticks along each line or, preferably, by small dots as shown.

The lengths and spacings of the lines may be varied to meet different conditions. In the example shown, the lines are 4 in. long, and the horizontal distances between them at the top and bottom are 1.80 in. and 2.20 in., respectively. Hence the distance at the middle is 2.00 in., and at all other points, as indicated on the scale. The left scale only is numbered.

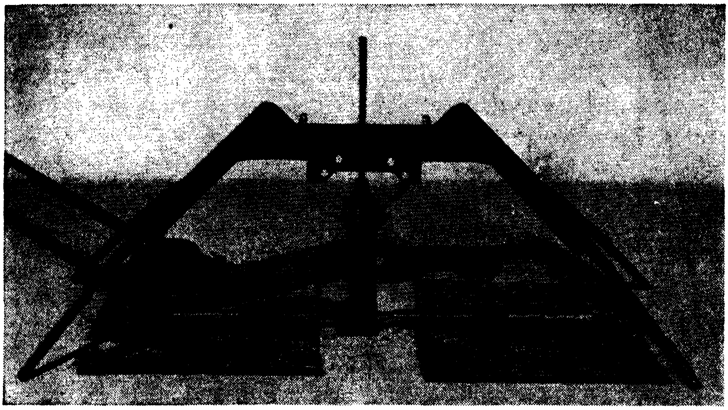
To use the wedge, the photographs should be oriented with the distance between the conjugate points about equal to the separation of the mid-point of the scale. Then under a stereoscope, where the wedge is placed so that a given indicated horizontal line is parallel with the line of flight, the dots near the ground should become fused and definite. When fusion has been effected, a line of dots will appear steeply inclined and will contact the ground at some dot on the numbered scale. If the wedge is then moved to another ground point and adjusted until some other numbered dot is in contact with the ground, the difference between the values of the two numbered dots is the measure of the parallax difference between the images of the two ground points. This difference in parallax determines the difference in elevation between the two ground points (see Eqs. **16-9** and 16-15).

The parallax wedge is inexpensive, convenient to use, and an excellent device to acquaint the student with the theory and use of parallax measurements.

16-41. The Parallax Bar Article 16-30 states that the parallax bar is an instrument with which the difference in parallax, Δp, may be measured with some precision. Such an instrument is illustrated in Fig. 16-23 and is discussed briefly in the following paragraphs.

The measurement of Δp is made by means of a micrometer which moves (parallel with the line of flight) one tiny dot with respect to the other. Each of the two dots is in the center of a separate glass disc, both of which are mounted on a bar. When these two dots are viewed properly under a stereoscope they fuse into a single dot called the *floating mark*. As the right-hand dot is moved toward the left one, the floating mark appears to move vertically downward toward the ground or below the ground surface. Also, as the right-hand dot is moved to the right, the floating mark will appear to move vertically upward. Hence, if the floating mark is apparently placed on the ground at a known elevation and the micrometer scale is read, and is then moved to another point of unknown elevation and the

micrometer turned until the floating mark again apparently rests on the ground surface, the difference in the two micrometer readings is a measure of Δp, and therefore a measure of the difference in elevation between the two points.



Courtesy of The Fairchild Camera and Instrument Corporation

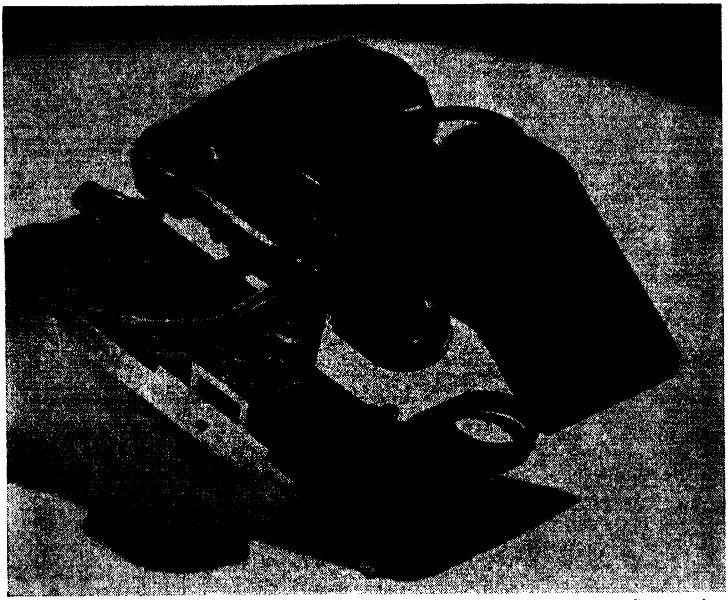
FIG. 16-23. The Parallax Bar.

If the parallax bar is attached to a drafting arm, and if a tracing point is also provided, as shown in the figure, it is possible to draw a planimetric map of the roads, streams, and other features in the stereoscopic view. Also, as stated above, the elevations of various points in the overlap area of the pair of photographs can be determined. Moreover, if the floating mark is set at a given elevation and then moved about, always just at the surface of the ground, a so-called *form* line is drawn. Such a line is a contour line in perspective and with proper subsequent treatment can be transformed into a correct contour line; thus a topographic map can be drawn.

The parallax bar is an instrument of relatively low precision and can hardly be considered capable of drawing a reliable map. It is more useful as an instrument for measuring spot elevations, for viewing the character of a terrain, and for making rough reconnaissance measurements. It is sometimes provided with binocular lenses to magnify further the field of view.

16-42. The Stereocomparagraph The stereocomparagraph, shown in Fig. 16-24, is a stereoscopic plotting instrument which

combines the features of the parallax bar, stereoscope, and tracing arm in one instrument and is capable of measuring spot elevations and of drawing planimetric features and form lines with greater precision than can be done with the parallax bar apparatus. Thus,



Courtesy of The Fairchild Camera and Instrument Corporation

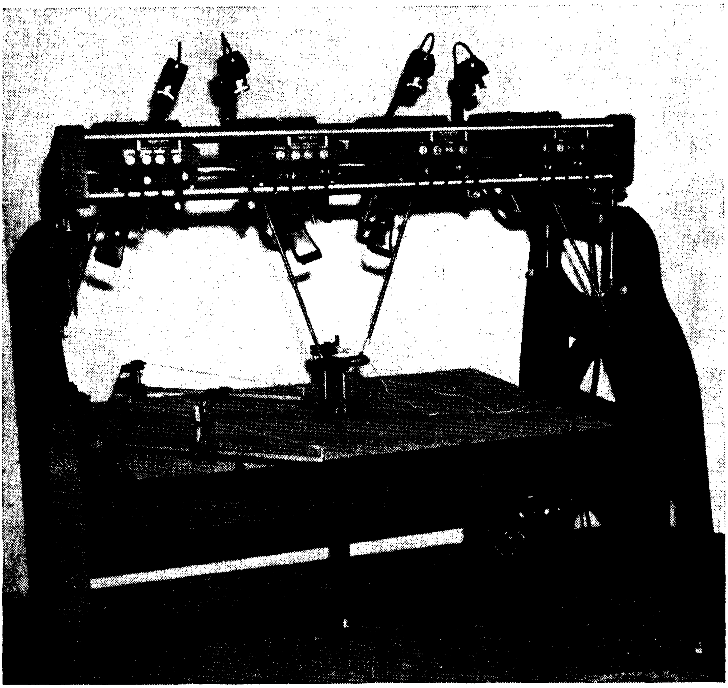
FIG. 16-24. The Stereocomparagraph.

the instrument provides a comparatively inexpensive and rapid means of constructing with some accuracy either a planimetric or a topographic map from vertical aerial photographs. The instrument is designed to be attached to a standard drafting arm.

With this instrument under good conditions, spot elevations can be measured with an average error of about one five-hundredth of the height of lens. Thus, if the height of lens is 10,000 ft, the error in the measured elevation of a point would be ± 20 ft. The instrument possesses the advantage that it can be operated by comparatively inexperienced draftsmen, and many operators can work simultaneously for the rapid completion of a project.

16-43. The Kelsh Plotter and the Autograph A-7 The Kelsh plotter is a precision mapping instrument which makes use of many

of the same principles as the multiplex projector. Its principal feature is the large size of the diapositive plates which eliminates the loss of accuracy inherent in the reduction printer of the multiplex instrument. The optical and mechanical properties of this instrument permit very definite control of the floating mark, which results in highly accurate maps (Fig. 16-25). The relatively low cost and satis-



Courtesy of Kelsh Instrument Company, Inc.

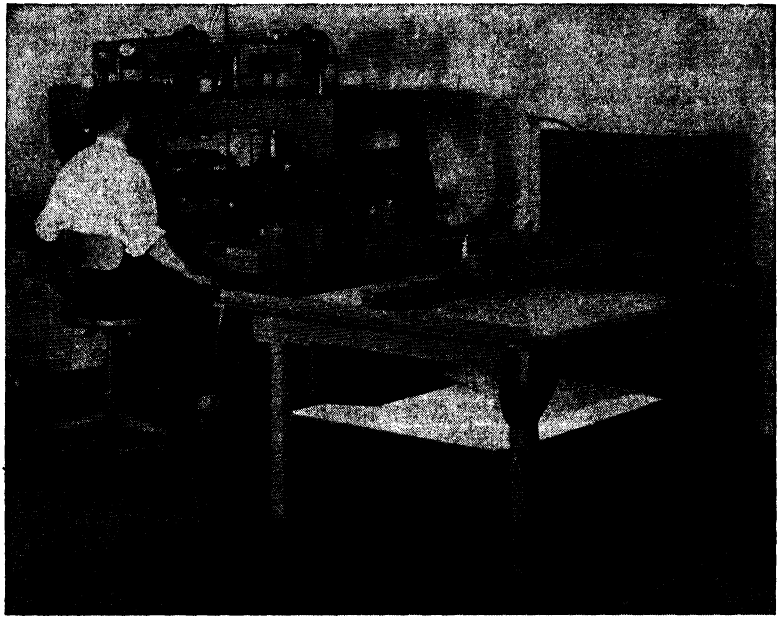
FIG. 16-25. The Kelsh Plotter, Model K-4.

factory performance of this instrument have earned for it a wide usage.

A number of still more elaborate and universally applicable photogrammetric plotting instruments have been designed. One of these is the Autograph A-7 model (Fig. 16-26), designed and manufactured by the Wild Company of Heerbrugg, Switzerland. Instruments of this kind create a view of a floating mark in a true spatial model of the terrain photographed. Either vertical, oblique, or horizontal pictures may be used. The movements of the floating mark in the

three dimensions of the model are effected by three controls simultaneously by the operator. The movements of the mark are translated by extremely precise mechanical devices to the drawing table where the map is drawn to the desired scale.

The two instruments described above may well be used in combination. The Autograph instrument may be used to extend both



Courtesy of Wild Heerbrugg Instruments, Inc.

FIG. 16-26. The Autograph A-7.

the vertical and horizontal map control (bridging) between ground-control stations. Then the photographs, thus controlled, can be placed in the Kelsh plotter, where the map is drawn. Thus, several relatively low-cost plotters may be used in combination with the Autograph to speed the work and reduce the total cost.

Any description of precision photogrammetric instruments should include some reference to the Brock stereometer and camera. These instruments and the accompanying methods were among the earliest to be devised and are the only ones that have been completely developed within the United States.

In the camera, glass plates were used, instead of film, to obtain

the negative. The contact negatives were then rephotographed to rectify the photographs and bring them to a uniform scale. These plates were then placed in the stereometer where the contours were drawn directly on the plates. Although this method is not now used, it exemplifies workmanship and principles that retain the highest respect of all photogrammetric engineers.

16-44. Uses and Accuracy of Photogrammetric Maps Early photogrammetric maps were drawn at small scales from terrestrial photographs, and their use was largely limited to broad general studies of natural resources.

When aerial photography became practical, mosaics and oblique photographs became immediately useful for military purposes, and they were also applied by many governmental and private agencies to a wide variety of uses.

With the development of precision stereoscopic instruments and methods, photogrammetric maps have been increasingly employed to include every use to which engineering or military maps were previously applied. Because of the added information contained in the photographs, many uses not previously possible are proving to be highly valuable, notably in the petroleum and forestry industries and by political taxing bodies.

The methods are now comparable in accuracy with the best ground methods, both for planimetric and topographic maps, of either flat or hilly terrain, and at either small or large scales. In very flat regions it may still be advantageous to use a plane table; but otherwise, wherever the survey is of sufficient size to warrant the use of aerial methods, a better map can be obtained by photogrammetric methods. Highway, municipal, and other surveys using scales as large as 1 in. = 100 ft and contour intervals of 1, 2, or 5 ft are being executed by private agencies in a fraction (as low as one-fifth) of the time and expense required by ground survey parties.

An important advantage of a photogrammetric map is consistent accuracy in every portion of the area. Ground surveys are subject to accumulative errors, mistakes, and limitations imposed by the terrain, such that different areas of the same map will show wide ranges in accuracy; but a photogrammetric map is the result of mechanical processes with checks and adjustments applied so that a common accuracy prevails throughout.

Office Problems

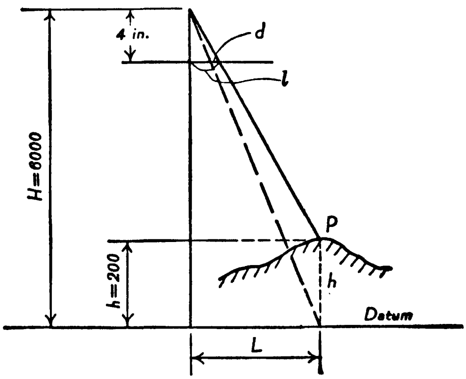
16-1. (a) If the representative fraction R of a photograph is 1/31,680, what is the scale factor S? (b) If $S = 660$, what is R?

16-2. If the focal length of a lens is $8\frac{1}{4}$ in., at what height must the airplane fly to yield a photograph for which $R = 1/24{,}000$? $S = 2640$?

16-3. An aerial survey is to be made under the following conditions: The area is 60 miles long by 24 miles wide; the negatives are 7 in. (in the direction of the line of flight) by 9 in.; $S = 2000$ ft per in.; forward lap = 60%; side lap = 30%. Compute the number of photographs required (a) when the flight strip is 60 miles, and (b) when the flight strip is 24 miles.

16-4. Given: $f = 4$ in.
$H = 6000$ ft
$l = 3.500$ in.

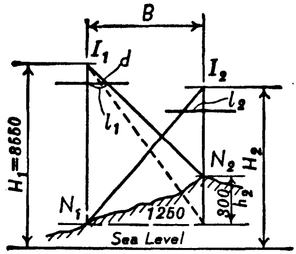
Find: S_o, S_p, d, and l.



PROB. 16-4.

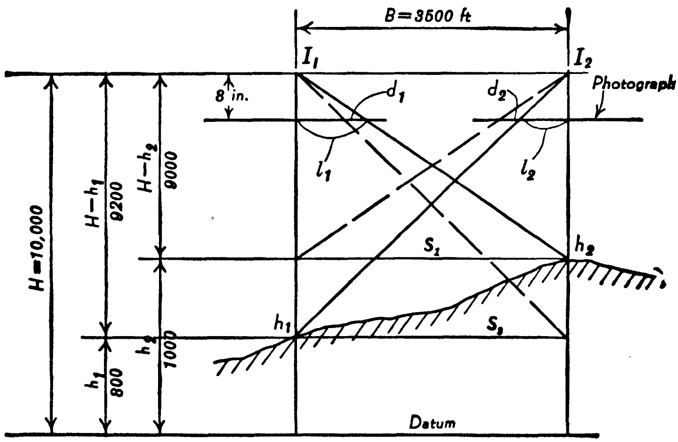
16-5. Given: $f = 4$ in.
$S_2 = 1800$ (at 1250 elev.)
$H_1 = 8550$ ft
$l_2 = 2.500$ in.
$h_1 = 1250$ ft
$h_2 = 300$ ft

Find: H_2, B, l_1, d.



PROB. 16-5.

16-6. Given the data shown, compute: (1) the scale S_1 of photograph I_1 at the elevation h_2; (2) the scale S_2 of photograph I_2 at the elevation h_1; (3) the photograph distances l_1 and l_2, and (4) the displacements d_1 and d_2.



PROB. 16-6.

16-7. Find the true distance between two points A and B, for which the following data are known: Height of lens = 6850 ft above sea level; elev. of A = 820 ft; elev. of B = 710 ft; f = 4.00 in.; the scaled coordinates on the photograph of the images a and b are x_a = 0.000 in., y_a = +0.93 in., x_b = +0.39 in., and y_b = −2.08in.

16-8. Given the data of Prob. 16-7 except that the length of the line AB is known to be 4634 ft, and the height of lens is assumed to be 7000 ft. Find the exact height of the lens.

16-9. Refer to Fig. 16-16. Given the following data: H = 16,000 ft; h_1 = 1200 ft; h_2 = 1450 ft; f = 8 in.; B = 5600 ft; l_1 = 2.040 in.;

$l_2 = 0.987$ in. Find: p_1, p_2, d_1, d_2, and show that $\Delta p = d_1 + d_2$. Also, find Δh from Eq. 16-11.

16-10. Find Δh for Prob. 16-9 by Eqs. (16-9) and (16-15).

16-11. Find Δh for Prob. 16-9 by the parallax-table method.

Answers to Office Problems

Prob. 16-1 (a) $S = 2640$, or 1 in. = 2640 ft (b) $R = \dfrac{1}{7920}$

Prob. 16-2 (a) 16,500 ft (b) 21,780 ft

Prob. 16-3 (a) 627 (b) 598

Prob. 16-4
$S_o = 1500$ ft per in.
$S_p = 1450$ ft per in.
$d = 0.117$ in.
$L = 5075$ ft

Prob. 16-5
$H_2 = 8450$ ft
$B = 4500$ ft
$l_1 = 2.571$ in.
$d = 0.106$ in.

Prob. 16-6
$S_1 = 1150$ ft per in.
$S_2 = 1125$ ft per in.
$l_1 = 3.111$ in.
$l_2 = 3.043$ in.
$d_1 = d_2 = 0.068$ in.

Prob. 16-7
$D = 4634$ ft

Prob. 16-8
$H = 6833$ ft

Prob. 16-9
$p_1 = 3.027$ in.
$p_2 = 3.079$ in.
$d_1 = 0.0345$ in.
$d_2 = 0.0167$ in.
$\Delta h = 250$ ft

Prob. 16-10
$\Delta h = 250$ ft

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CHAPTER 17

CONSTRUCTION SURVEYING

17-1. Remarks *Construction surveying* is primarily concerned with the establishment of certain lines and grades that guide and control construction operations. In short, construction surveying deals with the transfer of design dimensions from the engineering drawings to the ground so that the project is built in the correct position and with the proper relationship between its component parts.

There are no principles involved in the practice of construction surveying that have not been treated in prior chapters of this book. However, the project engineer will occasionally find that the problems of layout will tax his ingenuity and he will be unable to discover a precedent for a successful solution either in a textbook or in his past experience. It will remain for him to assess the field problem as intelligently as possible and devise a solution that will be sufficiently accurate and economical.

It is emphasized that before any layout work is begun the engineer should examine the major stakes or monuments that will control construction operations. For example, it is of critical importance that benchmarks be checked to reveal possible disturbance. The reference ties (see Art. 6-15) to key markers should be remeasured. Furthermore, it is essential to make certain that the correct elevation is used for any benchmark. An undetected mistake in the primary horizontal or vertical control could have serious consequences.

In addition to the preceding precautionary measures all tapes and leveling rods should be checked and levels and transits tested and, if necessary, adjusted.

Layout surveys take many forms and require various grades of accuracy in their execution. Included are staking operations for horizontal and vertical curves, land subdivisions, buildings, sewers, earthwork, and bridges. Only the essential features of a few representative types will be treated in this chapter.

17-2. Construction Grid System On major construction projects, such as at an industrial plant site, it is customary to establish a *construction grid system* to facilitate layout surveys and the recovery of important reference points which may become lost or disturbed. The grid axes are straight lines exactly at right angles to each other (see Art. 17-3) and should have their ends securely monumented with heavy poured-in-place concrete posts with metal tablets imbedded in their tops. Along both axes are set stout 2″ × 2″ tacked stakes at intervals of exactly 100 ft. The station values assigned to the west and south ends of the axes, which are preferably aligned with the cardinal directions, are large enough so that there is no possibility of negative station values being developed if the survey is extended to the south or west. In order to determine the relationship between the construction grid system coordinates and those of the state-wide plane coordinate system (see Art. 7-9), a line of connecting transit-traverse is run to the nearest government traverse or triangulation stations.

It is highly desirable that the elevation, preferably on the 1929 Mean Sea Level Datum, be determined for several strategically situated points in the construction area. This is particularly important when the industrial plant will have a complex of underground utilities which must be set correctly in elevation.

17-3. Precise Establishment of an Angle In staking out structures, setting column footings, bridge abutments, etc., it is frequently necessary to lay off predetermined angles with precision. It may be supposed that angle *COD* is to be established in the field, that line *OC* is fixed, and that the transit is in position at *O*. The *A* vernier is set at 0°, the instrument is sighted on *C*, the upper motion is released, and angle *COD* is set with the *A* vernier. A temporary point *D′*, which is approximately correct, is now set on this line. Then angle *COD′* is measured by repetition.

The difference between *COD′* and *COD* is then a small angle to be established by a linear measurement from *D′*, perpendicular to

OD' and equal to the tangent of the angle DOD' times the distance OD.

17-4. Slope Stakes If earthwork is to conform to a given alinement and side slope it is necessary to set slope stakes to guide the contractor in his work. Thus, at a given cross section for a roadway, slope stakes are set where the proposed side slopes meet the ground surface, as illustrated in Fig. 17-1.

If, following the execution of cross-section levels (see Art. 3-33), all cross sections have been plotted and the design cross section of the proposed roadway superimposed (see Fig. 3-25), it is necessary merely to scale the horizontal distance from the centerline to the intersection of the side slopes with the original ground surface and subsequently to lay off this distance in the field with a tape. However, slope stakes can also be located without such plotted cross sections by a trial-and-error procedure which will now be explained.

Fig. 17-1 shows the conditions for a roadway in excavation, 24 ft wide, with side slopes to the ratio of $1\frac{1}{2}$ (horizontal) to 1 (vertical). The height of instrument has been found by the usual process of differential leveling as indicated in the notes of Fig. 3-17. The elevation of subgrade is determined by the position of the grade line of the roadway as fixed from the profile study by the engineer.

The field party consists of four men: the engineer who supervises the work generally and who may keep the notes; a levelman; a rodman who, in addition to his rod, carries a 50-ft metallic tape; and a stakeman who aids the rodman in measuring distances and marks and drives stakes as directed.

It is more convenient to find the difference in elevation between the ground surface and the subgrade by means of rod readings than by computed elevations. This procedure makes use of a quantity called the "grade rod" which is found by subtracting the elevation of subgrade from the H.I., all rod readings and measured distances being taken to the nearest one tenth of a foot, only. Thus, the grade rod for the station shown is $637.4 - 626.1 = 11.3$. In other words, the grade rod is the reading that would be found on a rod if it could be held on the subgrade.

The *ground rod* at the center stake is 4.9 and obviously the cut is 6.4. This value is recorded in the notes as shown in Fig. 17-3; the numerator being the cut (or fill) and the denominator being the distance out d from the centerline to the slope stake. Of course, for the

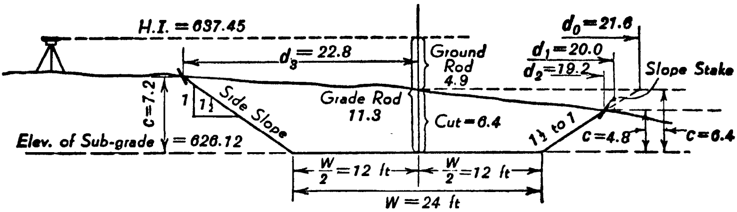


FIG. 17-1. Section in Cut.

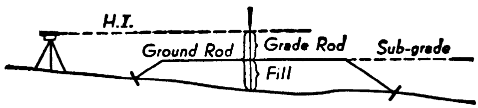


FIG. 17-2. Section in Fill.

SETTING SLOPE STAKES — ROUTE 47 — Buff Level No. 5 — July 25, 19 — Section — F.D. Park, ⊼ — F.R. Wood, Notes — S.M. Day, Tape — 22

Sta.	B.S.	H.I.	F.S.	Elev.	Grade	Grade Rod	L.	Center	R.
B.M.	5.04	637.45		632.41					
35					626.12	11.3	C 7.2 / 22.8	C 6.4 / 0	C 4.8 / 19.2
T.P.	2.43	629.64	10.24	627.21					
(A)+36					625.76	3.8	C 2.7 / 16.1	C 1.2 / 0	0 / 12.0
(H)+40					625.72	3.9			0 / 9.0
(I)+48					625.64	4.0		0 / 0	
(E)+56					625.56	4.0	0 / 9.0	F 0.8 / 0	F 2.4 / 12.6
(D)+58					625.54	4.1	0 / 12.0		
36					625.12	4.5	F 1.6 / 11.4	F 2.4 / 0	F 3.7 / 14.6
37					624.12	5.5	F 3.8 / 14.7	F 4.5 / 0	F 5.1 / 16.6
				etc.					

FIG. 17-3. Slope Stake Notes.

center stake, this distance is zero. The letter "C" is used to indicate a cut, and "F" a fill.

Having thus found the amount of cut at the center stake the party proceeds to locate a slope stake either to the right or left. On the right it is obvious that if the ground were level, as indicated by the dashed line, the distance to the slope stake would be $W/2 + 3/2 \times 6.4 = 12 + 9.6 = 21.6$ ft. But, as the rodman goes out from the center stake he notices that the ground slopes downward and that the position of the slope stake will be found at a distance less than 21.6 ft. Hence, he estimates a distance out d_1, say 20.0 ft, and holds his rod for a ground-rod reading. Suppose this reading is 6.5; then the indicated cut (grade-rod—ground-rod) is 4.8, for which the calculated distance out from the center, d_2, is $12 + 3/2 \times 4.8 = 19.2$ ft. But, the rodman estimated the distance and held the rod at a measured distance of 20.0 ft. There is a discrepancy, therefore, of 0.8 ft between the measured distance and the calculated distance from the centerline. Accordingly, the rod must be moved inward, and if the ground is assumed to be level for that small distance, the cut will be, as before, 4.8 ft and the correct location of the slope stake will be 19.2 ft. The record is entered as shown.

Sometimes it requires two or three trials before the correct location of the slope stake is found. But, it is always fixed at that point where the *measured* distance is equal to the *calculated* distance from the centerline.

In a similar manner the slope stake on the left side is found at a point where the cut (i.e., the distance above subgrade) is 7.2 ft and $d_3 = 22.8$ ft.

If the ground surface shows any marked change in slope between the centerline and the slope stake, additional readings are taken so that the correct area of the cross section can be computed. Such irregular sections will have four or more readings instead of three as shown.

The amount of the cut at the center is marked on the back of the center stake. Each slope stake has the cut (or fill) marked on the side facing the centerline and the station number on the back.

The procedure is similar in the case of a fill, except that the ground rod will be greater than the grade rod, and it is this condition that enables the party to determine whether any doubtful point marks a cut or a fill. See Fig. 17-2.

To avoid confusion it is always customary to subtract the ground

rod from the grade rod. If the result is positive, it represents a cut; if negative, it represents a fill.

Where the grade passes from cut to fill, it is necessary to set stakes at the points of zero cut and fill, i.e., at the points where excavation stops and fill begins. The roadway is wider in cuts than on fills, to provide side ditches for drainage. Hence, five grade points are usually located; two of which mark the edges of the cut, two mark the edges of the fill, and one the centerline.

If the ground surface transverse to the centerline is level, it is a simple matter to find the *grade points,* for they will all be located at the same station as that on the centerline. But if the ground surface transverse to the centerline is sloping, the location of the grade points is more complicated, as illustrated in Fig. 17-4.

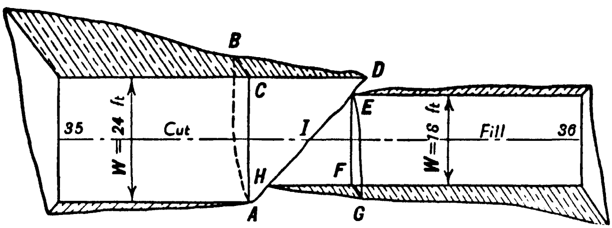


FIG. 17-4. Earthwork Sections in Cut and Fill.

The conditions assumed in the figure are that the roadway is 24 ft wide in cut and 18 ft wide on fill, and that the two end sections are at stations 35 and 36, respectively.

The five grade points are shown at A, H, I, E, and D. A and D mark the edges of the cut and are 12 ft from the centerline. H and E mark the edges of the fill and are 9 ft from the centerline. The station numbers of these points and the cross-section notes are shown in Fig. 17-3.

It may be noted that a cross section is taken at station A and at station E for the purpose of providing data to calculate the earthwork quantities. Thus, the excavation between station 35 and the grade points is found in two parts: (1) the prismoid having the two end sections at station 35 and at station $A = 35 + 36$, respectively; and (2) by the pyramid having the base ABC and the altitude CD. Likewise a cross section is taken at station $E = 35 + 56$, and the volume of the fill between the grade points and station 36 is found in two parts: (1) the pyramid having base EFG and altitude FH,

and (2) the prismoid having the two end sections at station $E = 35 + 56$ and at station 36, respectively.

The altitude CD of pyramid $ABCD$ is found from the station numbers in the notes to be 22 ft; the altitude FH of pyramid $EFGH = 16$ ft.

17-5. Grade Stakes In constructing any project to a given grade it is necessary that *grade stakes* be set to guide the contractor in his work. A grade stake is one driven until its top has the same elevation as the grade of the finished work, or until it has a known relation to that grade. In many cases, also, it fixes the alinement of the project. Examples are grade stakes for street pavements, sidewalks, sewers, railways, and highways.

1. *Street Pavements.* Grade stakes for street pavements are usually set outside (i.e., away from the centerline) of the curb about 2 or 3 ft, and are driven to fix the elevation of the top of the curb. The stakes are set at 50-ft intervals when the grade is uniform, and at 25-ft intervals on vertical curves, and are carefully set to fix the alinement of the back of the curb.

The level party then sets up the level in a convenient location, its H.I. being determined by differential levels from a nearby benchmark. The difference between the H.I. and the grade elevation of any given grade stake is the rod reading, or grade rod, for that stake. The stake is then driven down until, after repeated trials, the top of the stake has the desired elevation. Finally, the alinement is fixed on the stakes by tacks carefully lined in with a transit.

The tops of grade stakes are usually colored with red or blue keel to distinguish them from other stakes and to assure the contractor that they are at grade. If the location of the stakes is on a high bank such that much excavation would be necessary to set the stakes to grade, they may be set at a height of, say, 2 ft above grade. The contractor then measures down this amount to fix the elevation of the forms for the pavement.

2. *Sewers.* The grade line for a sewer is commonly established by fixing a line of sight, or by stretching a string, a known distance above the grade. At regular intervals of perhaps 50 ft along the centerline of the sewer, two stout stakes are driven, one on either side of the centerline. Having determined the grade rod for a given station, as indicated above, the rod is slid up or down along the stakes and a mark is made to establish the desired elevation. Then a cross

piece or "batter board" (see Fig. 17-5) is nailed to the two stakes, its top level and the elevation indicated by the mark. Thus, a series of batter boards are established and, if a string is stretched taut over the tops of these boards, it fixes a grade line at a known distance

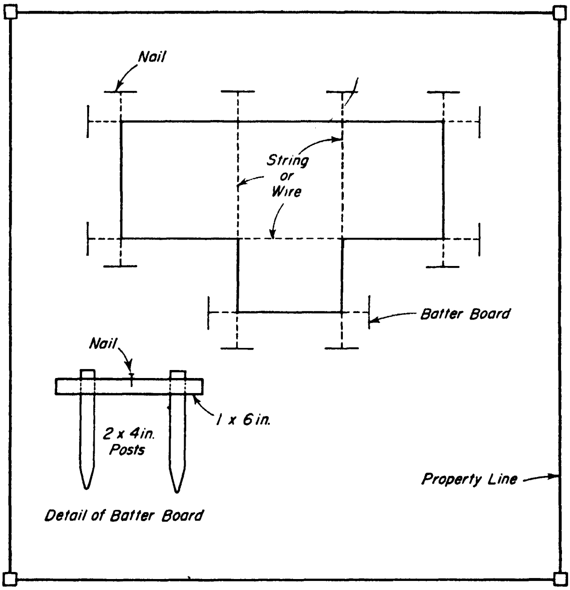


FIG. 17-5. Building Layout.

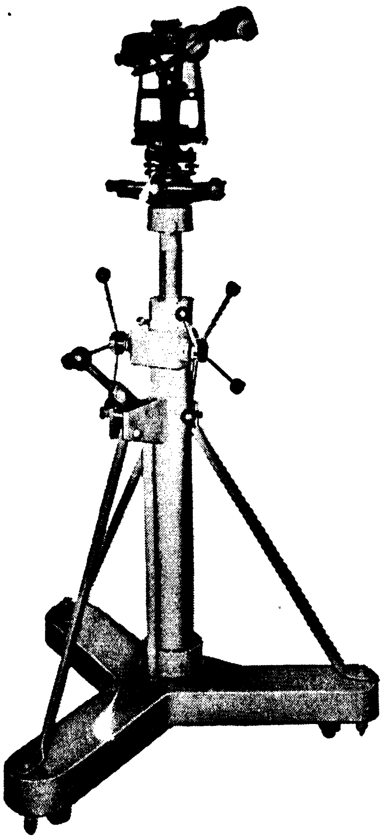
above the grade of the sewer. The workmen then measure this distance down from any point along the string to establish a point on the grade of the sewer.

3. *Railways.* Railway track is brought to its final grade by means of grade stakes. These are usually driven to the elevation of the top of rail and the track is then raised by tamping ballast underneath until the top of rail is level with the grade stakes.

17-6. Building Layout Before the lines of a building are established, it is frequently necessary to execute a property survey to locate the boundaries of the tract. Then the project engineer will proceed with the detailed location of the building with respect to the property lines.

Since stakes placed at the corners of the building would be con-

tinuously disturbed during excavation and construction operations, batter boards are erected. The upper edges of these boards are sometimes set at some special elevation such as that of the top of the foundation wall. Figure 17-5 shows that strings or wires connecting these boards serve to define certain building lines which are of great importance to the contractor. Nails driven into the top of the boards mark the exact position of the lines.



Keuffel & Esser Co.

FIG. 17-6. Stand for Jig Transit.

17-7. Optical Tooling Occasionally the engineer may be faced with the problem of providing very precise dimensional control in the erection and alignment of turbines, jigs, and other machine elements. The term *optical tooling* refers to surveying techniques that have been introduced into the aircraft and other industries in order to make accurate dimensional layouts possible. *Industrial surveying,* as it is also sometimes called, originally utilized the conventional transit and level in order to define lines and planes of reference in the shop. Modern industrial layouts and shop practices frequently permit dimensional tolerances of only a few thousandths of an inch. The stringency of these requirements led to the development of new instruments for conducting such work. Some are wholly new in design, whereas others, like the *jig transit,* represent modifications of conventional surveying equipment.

The jig transit (see Fig. 17-6) is designed especially for optical tooling. It is used principally to establish with precision vertical planes in industrial layout work. It differs from an ordinary transit

in that it has no horizontal or vertical circles, no compass, and only one horizontal motion. It is commonly mounted on a heavy metal stand. Since many shop sights are very short, the ability of the telescope to focus on a point as little as 3 ft from the instrument center is most essential.

Office Problems

17-1. In setting the column centers for a mill building it is necessary to turn a right angle with high precision. The angle is turned once and a point established at a distance of 110 ft. The angle is then repeated five times and the final reading of the vernier is 180°01′30″.

What distance perpendicular to the line of sight is to be measured to establish the right angle?

17-2. In setting slope stakes the following data are given: $W = 36$ ft, side slopes 1 to 1, cut at center stake = 4.3 ft, slope of ground of cross section is 4% downward from left to right. Find the amount of cut, and distance out to the slope stake (a) on the right and (b) on the left.

17-3. Given the following data: $W = 30$ ft, H.I. = 828.40; subgrade elev. = 825.16; side slope = $1\frac{1}{2}$ to 1; ground rod = 6.2; distance to rod from center = 20.0 ft.

(a) Is this section in cut or fill?
(b) Has the rodman found the right place for a slope stake?
(c) If not, which way should the rod be moved, and how far?

TABLE VII STADIA TABLE

Minutes	0° Hor. Dist.	0° Diff. Elev.	1° Hor. Dist.	1° Diff. Elev.	2° Hor. Dist.	2° Diff. Elev.	3° Hor. Dist.	3° Diff. Elev.
0	100.00	.00	99.97	1.74	99.88	3.49	99.73	5.23
2	100.00	.06	99.97	1.80	99.87	3.55	99.72	5.28
4	100.00	.12	99.97	1.86	99.87	3.60	99.71	5.34
6	100.00	.17	99.96	1.92	99.87	3.66	99.71	5.40
8	100.00	.23	99.96	1.98	99.86	3.72	99.70	5.46
10	100.00	.29	99.96	2.04	99.86	3.78	99.69	5.52
12	100.00	.35	99.96	2.09	99.85	3.84	99.69	5.57
14	100.00	.41	99.95	2.15	99.85	3.89	99.68	5.63
16	100.00	.47	99.95	2.21	99.84	3.95	99.68	5.69
18	100.00	.52	99.95	2.27	99.84	4.01	99.67	5.75
20	100.00	.58	99.95	2.33	99.83	4.07	99.66	5.80
22	100.00	.64	99.94	2.38	99.83	4.13	99.66	5.86
24	100.00	.70	99.94	2.44	99.82	4.18	99.65	5.92
26	99.99	.76	99.94	2.50	99.82	4.24	99.64	5.98
28	99.99	.81	99.93	2.56	99.81	4.30	99.63	6.04
30	99.99	.87	99.93	2.62	99.81	4.36	99.63	6.09
32	99.99	.93	99.93	2.67	99.80	4.42	99.62	6.15
34	99.99	.99	99.93	2.73	99.80	4.47	99.61	6.21
36	99.99	1.05	99.92	2.79	99.79	4.53	99.61	6.27
38	99.99	1.11	99.92	2.85	99.79	4.59	99.60	6.32
40	99.99	1.16	99.92	2.91	99.78	4.65	99.59	6.38
42	99.99	1.22	99.91	2.97	99.78	4.71	99.58	6.44
44	99.98	1.28	99.91	3.02	99.77	4.76	99.58	6.50
46	99.98	1.34	99.90	3.08	99.77	4.82	99.57	6.56
48	99.98	1.40	99.90	3.14	99.76	4.88	99.56	6.61
50	99.98	1.45	99.90	3.20	99.76	4.94	99.55	6.67
52	99.98	1.51	99.89	3.26	99.75	4.99	99.55	6.73
54	99.98	1.57	99.89	3.31	99.74	5.05	99.54	6.79
56	99.97	1.63	99.89	3.37	99.74	5.11	99.53	6.84
58	99.97	1.69	99.88	3.43	99.73	5.17	99.52	6.90
60	99.97	1.74	99.88	3.49	99.73	5.23	99.51	6.96

TABLE VII STADIA TABLE *(Continued)*

Minutes	4°		5°		6°		7°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	99.51	6.96	99.24	8.68	98.91	10.40	98.51	12.10
2	99.51	7.02	99.23	8.74	98.90	10.45	98.50	12.15
4	99.50	7.07	99.22	8.80	98.88	10.51	98.49	12.21
6	99.49	7.13	99.21	8.85	98.87	10.57	98.47	12.27
8	99.48	7.19	99.20	8.91	98.86	10.62	98.46	12.32
10	99.47	7.25	99.19	8.97	98.85	10.68	98.44	12.38
12	99.46	7.30	99.18	9.03	98.83	10.74	98.43	12.43
14	99.46	7.36	99.17	9.08	98.82	10.79	98.41	12.49
16	99.45	7.42	99.16	9.14	98.81	10.85	98.40	12.55
18	99.44	7.48	99.15	9.20	98.80	10.91	98.39	12.60
20	99.43	7.53	99.14	9.25	98.78	10.96	98.37	12.66
22	99.42	7.59	99.13	9.31	98.77	11.02	98.36	12.72
24	99.41	7.65	99.11	9.37	98.76	11.08	98.34	12.77
26	99.40	7.71	99.10	9.43	98.74	11.13	98.33	12.83
28	99.39	7.76	99.09	9.48	98.73	11.19	98.31	12.88
30	99.38	7.82	99.08	9.54	98.72	11.25	98.30	12.94
32	99.38	7.88	99.07	9.60	98.71	11.30	98.28	13.00
34	99.37	7.94	99.06	9.65	98.69	11.36	98.27	13.05
36	99.36	7.99	99.05	9.71	98.68	11.42	98.25	13.11
38	99.35	8.05	99.04	9.77	98.67	11.47	98.24	13.17
40	99.34	8.11	99.03	9.83	98.65	11.53	98.22	13.22
42	99.33	8.17	99.01	9.88	98.64	11.59	98.20	13.28
44	99.32	8.22	99.00	9.94	98.63	11.64	98.19	13.33
46	99.31	8.28	98.99	10.00	98.61	11.70	98.17	13.39
48	99.30	8.34	98.98	10.05	98.60	11.76	98.16	13.45
50	99.29	8.40	98.97	10.11	98.58	11.81	98.14	13.50
52	99.28	8.45	98.96	10.17	98.57	11.87	98.13	13.56
54	99.27	8.51	98.94	10.22	98.56	11.93	98.11	13.61
56	99.26	8.57	98.93	10.28	98.54	11.98	98.10	13.67
58	99.25	8.63	98.92	10.34	98.53	12.04	98.08	13.73
60	99.24	8.68	98.91	10.40	98.51	12.10	98.06	13.78

TABLE VII STADIA TABLE *(Continued)*

Minutes	8° Hor. Dist.	8° Diff. Elev.	9° Hor. Dist.	9° Diff. Elev.	10° Hor. Dist.	10° Diff. Elev.	11° Hor. Dist.	11° Diff. Elev.
0	98.06	13.78	97.55	15.45	96.98	17.10	96.36	18.73
2	98.05	13.84	97.53	15.51	96.96	17.16	96.34	18.78
4	98.03	13.89	97.52	15.56	96.94	17.21	96.32	18.84
6	98.01	13.95	97.50	15.62	96.92	17.26	96.29	18.89
8	98.00	14.01	97.48	15.67	96.90	17.32	96.27	18.95
10	97.98	14.06	97.46	15.73	96.88	17.37	96.25	19.00
12	97.97	14.12	97.44	15.78	96.86	17.43	96.23	19.05
14	97.95	14.17	97.43	15.84	96.84	17.48	96.21	19.11
16	97.93	14.23	97.41	15.89	96.82	17.54	96.18	19.16
18	97.92	14.28	97.39	15.95	96.80	17.59	96.16	19.21
20	97.90	14.34	97.37	16.00	96.78	17.65	96.14	19.27
22	97.88	14.40	97.35	16.06	96.76	17.70	96.12	19.32
24	97.87	14.45	97.33	16.11	96.74	17.76	96.09	19.38
26	97.85	14.51	97.31	16.17	96.72	17.81	96.07	19.43
28	97.83	14.56	97.29	16.22	96.70	17.86	96.05	19.48
30	97.82	14.62	97.28	16.28	96.68	17.92	96.03	19.54
32	97.80	14.67	97.26	16.33	96.66	17.97	96.00	19.59
34	97.78	14.73	97.24	16.39	96.64	18.03	95.98	19.64
36	97.76	14.79	97.22	16.44	96.62	18.08	95.96	19.70
38	97.75	14.84	97.20	16.50	96.60	18.14	95.93	19.75
40	97.73	14.90	97.18	16.55	96.57	18.19	95.91	19.80
42	97.71	14.95	97.16	16.61	96.55	18.24	95.89	19.86
44	97.69	15.01	97.14	16.66	96.53	18.30	95.86	19.91
46	97.68	15.06	97.12	16.72	96.51	18.35	95.84	19.96
48	97.66	15.12	97.10	16.77	96.49	18.41	95.82	20.02
50	97.64	15.17	97.08	16.83	96.47	18.46	95.79	20.07
52	97.62	15.23	97.06	16.88	96.45	18.51	95.77	20.12
54	97.61	15.28	97.04	16.94	96.42	18.57	95.75	20.18
56	97.59	15.34	97.02	16.99	96.40	18.62	95.72	20.23
58	97.57	15.40	97.00	17.05	96.38	18.68	95.70	20.28
60	97.55	15.45	96.98	17.10	96.36	18.73	95.68	20.34

TABLE VII STADIA TABLE *(Continued)*

Minutes	12°		13°		14°		15°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	95.68	20.34	94.94	21.92	94.15	23.47	93.30	25.00
2	95.65	20.39	94.91	21.97	94.12	23.52	93.27	25.05
4	95.63	20.44	94.89	22.02	94.09	23.58	93.24	25.10
6	95.61	20.50	94.86	22.08	94.07	23.63	93.21	25.15
8	95.58	20.55	94.84	22.13	94.04	23.68	93.18	25.20
10	95.56	20.60	94.81	22.18	94.01	23.73	93.16	25.25
12	95.53	20.66	94.79	22.23	93.98	23.78	93.13	25.30
14	95.51	20.71	94.76	22.28	93.95	23.83	93.10	25.35
16	95.49	20.76	94.73	22.34	93.93	23.88	93.07	25.40
18	95.46	20.81	94.71	22.39	93.90	23.93	93.04	25.45
20	95.44	20.87	94.68	22.44	93.87	23.99	93.01	25.50
22	95.41	20.92	94.66	22.49	93.84	24.04	92.98	25.55
24	95.39	20.97	94.63	22.54	93.82	24.09	92.95	25.60
26	95.36	21.03	94.60	22.60	93.79	24.14	92.92	25.65
28	95.34	21.08	94.58	22.65	93.76	24.19	92.89	25.70
30	95.32	21.13	94.55	22.70	93.73	24.24	92.86	25.75
32	95.29	21.18	94.52	22.75	93.70	24.29	92.83	25.80
34	95.27	21.24	94.50	22.80	93.67	24.34	92.80	25.85
36	95.24	21.29	94.47	22.85	93.65	24.39	92.77	25.90
38	95.22	21.34	94.44	22.91	93.62	24.44	92.74	25.95
40	95.19	21.39	94.42	22.96	93.59	24.49	92.71	26.00
42	95.17	21.45	94.39	23.01	93.56	24.55	92.68	26.05
44	95.14	21.50	94.36	23.06	93.53	24.60	92.65	26.10
46	95.12	21.55	94.34	23.11	93.50	24.65	92.62	26.15
48	95.09	21.60	94.31	23.16	93.47	24.70	92.59	26.20
50	95.07	21.66	94.28	23.22	93.45	24.75	92.56	26.25
52	95.04	21.71	94.26	23.27	93.42	24.80	92.53	26.30
54	95.02	21.76	94.23	23.32	93.39	24.85	92.49	26.35
56	94.99	21.81	94.20	23.37	93.36	24.90	92.46	26.40
58	94.97	21.87	94.17	23.42	93.33	24.95	92.43	26.45
60	94.94	21.92	94.15	23.47	93.30	25.00	92.40	26.50

TABLE VII STADIA TABLE *(Continued)*

Minutes	16°		17°		18°		19°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	92.40	26.50	91.45	27.96	90.45	29.39	89.40	30.78
2	92.37	26.55	91.42	28.01	90.42	29.44	89.36	30.83
4	92.34	26.59	91.39	28.06	90.38	29.48	89.33	30.87
6	92.31	26.64	91.35	28.10	90.35	29.53	89.29	30.92
8	92.28	26.69	91.32	28.15	90.31	29.58	89.26	30.97
10	92.25	26.74	91.29	28.20	90.28	29.62	89.22	31.01
12	92.22	26.79	91.26	28.25	90.24	29.67	89.18	31.06
14	92.19	26.84	91.22	28.30	90.21	29.72	89.15	31.10
16	92.15	26.89	91.19	28.34	90.18	29.76	89.11	31.15
18	92.12	26.94	91.16	28.39	90.14	29.81	89.08	31.19
20	92.09	26.99	91.12	28.44	90.11	29.86	89.04	31.24
22	92.06	27.04	91.09	28.49	90.07	29.90	89.00	31.28
24	92.03	27.09	91.06	28.54	90.04	29.95	88.97	31.33
26	92.00	27.13	91.02	28.58	90.00	30.00	88.93	31.38
28	91.97	27.18	90.99	28.63	89.97	30.04	88.89	31.42
30	91.93	27.23	90.96	28.68	89.93	30.09	88.86	31.47
32	91.90	27.28	90.92	28.73	89.90	30.14	88.82	31.51
34	91.87	27.33	90.89	28.77	89.86	30.18	88.78	31.56
36	91.84	27.38	90.86	28.82	89.83	30.23	88.75	31.60
38	91.81	27.43	90.82	28.87	89.79	30.28	88.71	31.65
40	91.77	27.48	90.79	28.92	89.76	30.32	88.67	31.69
42	91.74	27.52	90.76	28.96	89.72	30.37	88.64	31.74
44	91.71	27.57	90.72	29.01	89.69	30.41	88.60	31.78
46	91.68	27.62	90.69	29.06	89.65	30.46	88.56	31.83
48	91.65	27.67	90.66	29.11	89.61	30.51	88.53	31.87
50	91.61	27.72	90.62	29.15	89.58	30.55	88.49	31.92
52	91.58	27.77	90.59	29.20	89.54	30.60	88.45	31.96
54	91.55	27.81	90.55	29.25	89.51	30.65	88.41	32.01
56	91.52	27.86	90.52	29.30	89.47	30.69	88.38	32.05
58	91.48	27.91	90.49	29.34	89.44	30.74	88.34	32.09
60	91.45	27.96	90.45	29.39	89.40	30.78	88.30	32.14

TABLE VII STADIA TABLE *(Continued)*

Minutes	20°		21°		22°		23°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	88.30	32.14	87.16	33.46	85.97	34.73	84.73	35.97
2	88.26	32.18	87.12	33.50	85.93	34.77	84.69	36.01
4	88.23	32.23	87.08	33.54	85.89	34.82	84.65	36.05
6	88.19	32.27	87.04	33.59	85.85	34.86	84.61	36.09
8	88.15	32.32	87.00	33.63	85.80	34.90	84.57	36.13
10	88.11	32.36	86.96	33.67	85.76	34.94	84.52	36.17
12	88.08	32.41	86.92	33.72	85.72	34.98	84.48	36.21
14	88.04	32.45	86.88	33.76	85.68	35.02	84.44	36.25
16	88.00	32.49	86.84	33.80	85.64	35.07	84.40	36.29
18	87.96	32.54	86.80	33.84	85.60	35.11	84.35	36.33
20	87.93	32.58	86.77	33.89	85.56	35.15	84.31	36.37
22	87.89	32.63	86.73	33.93	85.52	35.19	84.27	36.41
24	87.85	32.67	86.69	33.97	85.48	35.23	84.23	36.45
26	87.81	32.72	86.65	34.01	85.44	35.27	84.18	36.49
28	87.77	32.76	86.61	34.06	85.40	35.31	84.14	36.53
30	87.74	32.80	86.57	34.10	85.36	35.36	84.10	36.57
32	87.70	32.85	86.53	34.14	85.31	35.40	84.06	36.61
34	87.66	32.89	86.49	34.18	85.27	35.44	84.01	36.65
36	87.62	32.93	86.45	34.23	85.23	35.48	83.97	36.69
38	87.58	32.98	86.41	34.27	85.19	35.52	83.93	36.73
40	87.54	33.02	86.37	34.31	85.15	35.56	83.89	36.77
42	87.51	33.07	86.33	34.35	85.11	35.60	83.84	36.80
44	87.47	33.11	86.29	34.40	85.07	35.64	83.80	36.84
46	87.43	33.15	86.25	34.44	85.02	35.68	83.76	36.88
48	87.39	33.20	86.21	34.48	84.98	35.72	83.72	36.92
50	87.35	33.24	86.17	34.52	84.94	35.76	83.67	36.96
52	87.31	33.28	86.13	34.57	84.90	35.80	83.63	37.00
54	87.27	33.33	86.09	34.61	84.86	35.85	83.59	37.04
56	87.24	33.37	86.05	34.65	84.82	35.89	83.54	37.08
58	87.20	33.41	86.01	34.69	84.77	35.93	83.50	37.12
60	87.16	33.46	85.97	34.73	84.73	35.97	83.46	37.16

TABLE VII STADIA TABLE (Continued)

Minutes	24°		25°		26°		27°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	83.46	37.16	82.14	38.30	80.78	39.40	79.39	40.45
2	83.41	37.20	82.09	38.34	80.74	39.44	79.34	40.49
4	83.37	37.23	82.05	38.38	80.69	39.47	79.30	40.52
6	83.33	37.27	82.01	38.41	80.65	39.51	79.25	40.55
8	83.28	37.31	81.96	38.45	80.60	39.54	79.20	40.59
10	83.24	37.35	81.92	38.49	80.55	39.58	79.15	40.62
12	83.20	37.39	81.87	38.53	80.51	39.61	79.11	40.66
14	83.15	37.43	81.83	38.56	80.46	39.65	79.06	40.69
16	83.11	37.47	81.78	38.60	80.41	39.69	79.01	40.72
18	83.07	37.51	81.74	38.64	80.37	39.72	78.96	40.76
20	83.02	37.54	81.69	38.67	80.32	39.76	78.92	40.79
22	82.98	37.58	81.65	38.71	80.28	39.79	78.87	40.82
24	82.93	37.62	81.60	38.75	80.23	39.83	78.82	40.86
26	82.89	37.66	81.56	38.78	80.18	39.86	78.77	40.89
28	82.85	37.70	81.51	38.82	80.14	39.90	78.73	40.92
30	82.80	37.74	81.47	38.86	80.09	39.93	78.68	40.96
32	82.76	37.77	81.42	38.89	80.04	39.97	78.63	40.99
34	82.72	37.81	81.38	38.93	80.00	40.00	78.58	41.02
36	82.67	37.85	81.33	38.97	79.95	40.04	78.54	41.06
38	82.63	37.89	81.28	39.00	79.90	40.07	78.49	41.09
40	82.58	37.93	81.24	39.04	79.86	40.11	78.44	41.12
42	82.54	37.96	81.19	39.08	79.81	40.14	78.39	41.16
44	82.49	38.00	81.15	39.11	79.76	40.18	78.34	41.19
46	82.45	38.04	81.10	39.15	79.72	40.21	78.30	41.22
48	82.41	38.08	81.06	39.18	79.67	40.24	78.25	41.26
50	82.36	38.11	81.01	39.22	79.62	40.28	78.20	41.29
52	82.32	38.15	80.97	39.26	79.58	40.31	78.15	41.32
54	82.27	38.19	80.92	39.29	79.53	40.35	78.10	41.35
56	82.23	38.23	80.87	39.33	79.48	40.38	78.06	41.39
58	82.18	38.26	80.83	39.36	79.44	40.42	78.01	41.42
60	82.14	38.30	80.78	39.40	79.39	40.45	77.96	41.45

TABLE VII STADIA TABLE *(Continued)*

Minutes	28°		29°		30°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0	77.96	41.45	76.50	42.40	75.00	43.30
2	77.91	41.48	76.45	42.43	74.95	43.33
4	77.86	41.52	76.40	42.46	74.90	43.36
6	77.81	41.55	76 35	42.49	74.85	43.39
8	77.77	41.58	76.30	42.53	74.80	43.42
10	77.72	41.61	76.25	42.56	74.75	43.45
12	77.67	41.65	76.20	42.59	74.70	43.47
14	77.62	41.68	76.15	42.62	74.65	43.50
16	77.57	41.71	76.10	42.65	74.60	43.53
18	77.52	41.74	76.05	42.68	74.55	43.56
20	77.48	41.77	76.00	42.71	74.49	43.59
22	77.42	41.81	75.95	42.74	74.44	43.62
24	77.38	41.84	75.90	42.77	74.39	43.65
26	77.33	41.87	75.85	42.80	74.34	43.67
28	77.28	41.90	75.80	42.83	74.29	43.70
30	77.23	41.93	75.75	42.86	74.24	43.73
32	77.18	41.97	75.70	42.89	74.19	43.76
34	77.13	42.00	75.65	42.92	74.14	43.79
36	77.09	42.03	75.60	42.95	74.09	43.82
38	77.04	42.06	75.55	42.98	74.04	43.84
40	76.99	42.09	75.50	43.01	73.99	43.87
42	76.94	42.12	75.45	43.04	73.93	43.90
44	76.89	42.15	75.40	43.07	73.88	43.93
46	76.84	42.19	75.35	43.10	73.83	43.95
48	76.79	42.22	75.30	43.13	73.78	43.98
50	76.74	42.25	75.25	43.16	73.73	44.01
52	76.69	42.28	75.20	43.18	73.68	44.04
54	76.64	42.31	75.15	43.21	73.63	44.07
56	76.59	42.34	75.10	43.24	73.58	44.09
58	76.55	42.37	75.05	43.27	73.52	44.12
60	76.50	42.40	75.00	43.30	73.47	44.15

TABLE VIII RADII FOR CIRCULAR CURVES—Chord Definition

Deg. D.	Radius R.	Log. R.	Deg. D.	Radius R.	Log. R.	Deg. D.	Radius R.	Log. R.
° ′			° ′			° ′		
0 0	∞	∞	1 0	5729.65	3.758128	2 0	2864.93	3.457115
1	343774.68	5.536274	1	5635.72	.750950	1	2841.26	.453511
2	171887.34	.235244	2	5544.83	.743888	2	2817.97	.449937
3	114591.56	.059153	3	5456.82	.736939	3	2795.06	.446392
4	85943.67	4.934214	4	5371.56	.730100	4	2772.53	.442876
5	68754.94	4.837304	5	5288.92	3.723367	5	2750.35	3.439388
6	57295.79	.758123	6	5208.79	.716737	6	2728.52	.435928
7	49110.68	.691176	7	5131.05	.710206	7	2707.04	.432495
8	42971.84	.633184	8	5055.59	.703772	8	2685.89	.429089
9	38197.20	.582031	9	4982.33	.697432	9	2665.08	.425710
10	34377.48	4.536274	10	4911.15	3.691183	10	2644.58	3.422356
11	31252.26	.494881	11	4841.98	.685023	11	2624.39	.419029
12	28647.90	.457093	12	4774.74	.678949	12	2604.51	.415727
13	26444.22	.422331	13	4709.33	.672959	13	2584.93	.412449
14	24555.35	.390146	14	4645.69	.667051	14	2565.65	.409197
15	22918.33	4.360183	15	4583.75	3.661221	15	2546.64	3.405968
16	21485.94	.332154	16	4523.44	.655469	16	2527.92	.402763
17	20222.06	.305825	17	4464.70	.649792	17	2509.47	.399582
18	19098.61	.281002	18	4407.46	.644189	18	2491.29	.396424
19	18093.43	.257521	19	4351.67	.638656	19	2473.37	.393289
20	17188.76	4.235244	20	4297.28	3.633194	20	2455.70	3.390176
21	16370.25	.214055	21	4244.23	.627799	21	2438.29	.387085
22	15626.15	.193852	22	4192.47	.622470	22	2421.12	.384016
23	14946.75	.174547	23	4141.96	.617206	23	2404.19	.380969
24	14323.97	.156064	24	4092.66	.612005	24	2387.50	.377943
25	13751.02	4.138335	25	4044.51	3.606866	25	2371.04	3.374938
26	13222.13	.121302	26	3997.48	.601787	26	2354.80	.371954
27	12732.43	.104911	27	3951.54	.596766	27	2338.78	.368990
28	12277.70	.089117	28	3906.64	.591803	28	2322.98	.366046
29	11854.33	.073877	29	3862.74	.586896	29	2307.39	.363122
30	11459.19	4.059154	30	3819.83	3.582044	30	2292.01	3.360217
31	11089.54	.044914	31	3777.85	.577245	31	2276.84	.357332
32	10743.00	.031125	32	3736.79	.572499	32	2261.86	.354466
33	10417.45	.017762	33	3696.61	.567804	33	2247.08	.351618
34	10111.06	.004797	34	3657.29	.563160	34	2232.49	.348789
35	9822.18	3.992208	35	3618.80	3.558564	35	2218.09	3.345979
36	9549.34	.979973	36	3581.10	.554017	36	2203.87	.343187
37	9291.25	.968074	37	3544.19	.549517	37	2189.84	.340412
38	9046.75	.956492	38	3508.02	.545063	38	2175.98	.337655
39	8814.78	.945212	39	3472.59	.540654	39	2162.30	.334915
40	8594.42	3.934216	40	3437.87	3.536289	40	2148.79	3.332193
41	8384.80	.923493	41	3403.83	.531968	41	2135.44	.329488
42	8185.16	.913027	42	3370.46	.527690	42	2122.26	.326799
43	7994.81	.902808	43	3337.74	.523453	43	2109.24	.324127
44	7813.11	.892824	44	3305.65	.519257	44	2096.39	.321471
45	7639.49	3.883064	45	3274.17	3.515101	45	2083.68	3.318832
46	7473.42	.873519	46	3243.29	.510985	46	2071.13	.316208
47	7314.41	.864179	47	3212.98	.506908	47	2058.73	.313600
48	7162.03	.855036	48	3183.23	.50[illegible]868	48	2046.48	.311008
49	7015.87	.846081	49	3154.03	.498866	49	2034.37	.308431
50	6875.55	3.837308	50	3125.36	3.494900	50	2022.41	3.305869
51	6740.74	.828708	51	3097.20	.490970	51	2010.59	.303323
52	6611.12	.820275	52	3069.55	.487075	52	1998.90	.300791
53	6486.38	.812002	53	3042.39	.483215	53	1987.35	.298274
54	6366.26	.803885	54	3015.71	.479389	54	1975.93	.295771
55	6250.51	3.795916	55	2989.48	3.475596	55	1964.64	3.293283
56	6138.90	.788091	56	2963.72	.471836	56	1953.48	.290809
57	6031.20	.780404	57	2938.39	.468109	57	1942.44	.288349
58	5927.22	.772851	58	2913.49	.464413	58	1931.53	.285902
59	5826.76	.765427	59	2889.01	.460749	59	1920.75	.283470
60	5729.65	3.758128	60	2864.93	3.457115	60	1910.08	3.281051

Deg. D. ° ′	Radius R.	Log. R.	Deg. D. ° ′	Radius R.	Log. R.	Deg. D. ° ′	Radius R.	Log. R.
3 0	1910.08	3.281051	4 0	1432.69	3.156151	5 0	1146.28	3.059290
1	1899.53	.278645	1	1426.74	.154346	1	1142.47	.057846
2	1889.09	.276253	2	1420.85	.152548	2	1138.69	.056407
3	1878.77	.273874	3	1415.01	.150758	3	1134.94	.054972
4	1868.56	.271508	4	1409.21	.148975	4	1131.21	.053542
5	1858.47	3.269155	5	1403.46	3.147200	5	1127.50	3.052116
6	1848.48	.266814	6	1397.76	.145431	6	1123.82	.050696
7	1838.59	.264486	7	1392.10	.143670	7	1120.16	.049280
8	1828.82	.262170	8	1386.49	.141916	8	1116.52	.047868
9	1819.14	.259867	9	1380.92	.140169	9	1112.91	.046462
10	1809.57	3.257576	10	1375.40	3.138430	10	1109.33	3.045059
11	1800.10	.255296	11	1369.92	.136697	11	1105.76	.043662
12	1790.73	.253029	12	1364.49	.134971	12	1102.22	.042268
13	1781.45	.250774	13	1359.10	.133251	13	1098.70	.040880
14	1772.27	.248530	14	1353.75	.131539	14	1095.20	.039495
15	1763.18	3.246297	15	1348.45	3.129833	15	1091.73	3.038115
16	1754.19	.244077	16	1343.18	.128134	16	1088.28	.036740
17	1745.29	.241867	17	1337.96	.126442	17	1084.85	.035368
18	1736.48	.239669	18	1332.77	.124756	18	1081.44	.034002
19	1727.75	.237481	19	1327.63	.123077	19	1078.05	.032639
20	1719.12	3.235305	20	1322.53	3.121404	20	1074.68	3.031281
21	1710.57	.233140	21	1317.46	.119738	21	1071.34	.029927
22	1702.10	.230985	22	1312.43	.118078	22	1068.01	.028577
23	1693.72	.228841	23	1307.45	.116424	23	1064.71	.027231
24	1685.42	.226707	24	1302.50	.114777	24	1061.43	.025890
25	1677.20	3.224584	25	1297.58	3.113136	25	1058.16	3.024552
26	1669.06	.222472	26	1292.71	.111501	26	1054.92	.023219
27	1661.00	.220369	27	1287.87	.109872	27	1051.70	.021890
28	1653.02	.218277	28	1283.07	.108249	28	1048.49	.020565
29	1645.11	.216194	29	1278.30	.106632	29	1045.31	.019244
30	1637.28	3.214122	30	1273.57	3.105022	30	1042.14	3.017927
31	1629.52	.212060	31	1268.87	.103417	31	1039.00	.016614
32	1621.84	.210007	32	1264.21	.101818	32	1035.87	.015305
33	1614.22	.207964	33	1259.58	.100225	33	1032.76	.013999
34	1606.68	.205930	34	1254.98	.098638	34	1029.67	.012698
35	1599.21	3.203906	35	1250.42	3.097057	35	1026.60	3.011401
36	1591.81	.201892	36	1245.89	.095481	36	1023.55	.010107
37	1584.48	.199886	37	1241.40	.093912	37	1020.51	.008818
38	1577.21	.197890	38	1236.94	.092347	38	1017.49	.007532
39	1570.01	.195903	39	1232.51	.090789	39	1014.50	.006250
40	1562.88	3.193925	40	1228.11	3.089236	40	1011.51	3.004972
41	1555.81	.191956	41	1223.74	.087688	41	1008.55	.003698
42	1548.80	.189996	42	1219.40	.086147	42	1005.60	.002427
43	1541.86	.188045	43	1215.09	.084610	43	1002.67	.001160
44	1534.98	.186103	44	1210.82	.083079	44	999.76	2.999897
45	1528.16	3.184169	45	1206.57	3.081553	45	996.87	2.998637
46	1521.40	.182244	46	1202.36	.080033	46	993.99	.997381
47	1514.70	.180327	47	1198.17	.078518	47	991.13	.996129
48	1508.06	.178419	48	1194.01	.077008	48	988.28	.994880
49	1501.48	.176519	49	1189.88	.075504	49	985.45	.993635
50	1494.95	3.174627	50	1185.78	3.074005	50	982.64	2.992393
51	1488.48	.172744	51	1181.71	.072511	51	979.84	.991155
52	1482.07	.170868	52	1177.66	.071022	52	977.06	.989921
53	1475.71	.169001	53	1173.65	.069538	53	974.29	.988690
54	1469.41	.167142	54	1169.66	.068059	54	971.54	.987463
55	1463.16	3.165291	55	1165.70	3.066585	55	968.81	2.986239
56	1456.96	.163447	56	1161.76	.065116	56	966.09	.985018
57	1450.81	.161612	57	1157.85	.063653	57	963.39	.983801
58	1444.72	.159784	58	1153.97	.062194	58	960.70	.982587
59	1438.68	.157963	59	1150.11	.060740	59	958.02	.981377
60	1432.69	3.156151	60	1146.28	3.059290	60	955.37	2.980170

TABLE IX TANGENTS AND EXTERNALS TO A 1° CURVE* —Chord Definition

Angle	Tangent	External	Angle	Tangent	External	Angle	Tangent	External
1° 00′	50.00	.22	11° 00′	551.70	26.50	21° 00′	1061.9	97.57
10	58.34	.30	10	560.11	27.31	10	1070.6	99.16
20	66.67	.39	20	568.53	28.14	20	1079.2	100.75
30	75.01	.49	30	576.95	28.97	30	1087.8	102.35
40	83.34	.61	40	585.36	29.82	40	1096.4	103.97
50	91.68	.73	50	593.79	30.68	50	1105.1	105.60
2° 00′	100.01	.87	12° 00′	602.21	31.56	22° 00′	1113.7	107.24
10	108.35	1.02	10	610.64	32.45	10	1122.4	108.90
20	116.68	1.19	20	619.07	33.35	20	1131.0	110.57
30	125.02	1.36	30	627.50	34.26	30	1139.7	112.25
40	133.36	1.55	40	635.93	35.18	40	1148.4	113.95
50	141.70	1.75	50	644.37	36.12	50	1157.0	115.66
3° 00′	150.04	1.96	13° 00′	652.81	37.07	23° 00′	1165.7	117.38
10	158.38	2.19	10	661.25	38.03	10	1174.4	119.12
20	166.72	2.43	20	669.70	39.01	20	1183.1	120.87
30	175.06	2.67	30	678.15	39.99	30	1191.8	122.63
40	183.40	2.93	40	686.60	40.99	40	1200.5	124.41
50	191.74	3.21	50	695.06	42.00	50	1209.2	126.20
4° 00′	200.08	3.49	14° 00′	703.51	43.03	24° 00′	1217.9	128.00
10	208.43	3.79	10	711.97	44.07	10	1226.6	129.82
20	216.77	4.10	20	720.44	45.12	20	1235.3	131.65
30	225.12	4.42	30	728.90	46.18	30	1244.0	133.50
40	233.47	4.76	40	737.37	47.25	40	1252.8	135.35
50	241.81	5.10	50	745.85	48.34	50	1261.5	137.23
5° 00′	250.16	5.46	15° 00′	754.32	49.44	25° 00′	1270.2	139.11
10	258.51	5.83	10	762.80	50.55	10	1279.0	141.01
20	266.86	6.21	20	771.29	51.68	20	1287.7	142.93
30	275.21	6.61	30	779.77	52.89	30	1296.5	144.85
40	283.57	7.01	40	788.26	53.97	40	1305.3	146.79
50	291.92	7.43	50	796.75	55.13	50	1314.0	148.75
6° 00′	300.28	7.86	16° 00′	805.25	56.31	26° 00′	1322.8	150.71
10	308.64	8.31	10	813.75	57.50	10	1331.6	152.69
20	316.99	8.76	20	822.25	58.70	20	1340.4	154.69
30	325.35	9.23	30	830.76	59.91	30	1349.2	156.70
40	333.71	9.71	40	839.27	61.14	40	1358.0	158.72
50	342.08	10.20	50	847.78	62.38	50	1366.8	160.76
7° 00′	350.44	10.71	17° 00′	856.30	63.63	27° 00′	1375.6	162.81
10	358.81	11.22	10	864.82	64.90	10	1384.4	164.86
20	367.17	11.75	20	873.35	66.18	20	1393.2	166.95
30	375.54	12.29	30	881.88	67.47	30	1402.0	169.04
40	383.91	12.85	40	890.41	68.77	40	1410.9	171.15
50	392.28	13.41	50	898.95	70.09	50	1419.7	173.27
8° 00′	400.66	13.99	18° 00′	907.49	71.42	28° 00′	1428.6	175.41
10	409.03	14.58	10	916.03	72.76	10	1437.4	177.55
20	417.41	15.18	20	924.58	74.12	20	1446.3	179.72
30	425.79	15.80	30	933.13	75.49	30	1455.1	181.89
40	434.17	16.43	40	941.69	76.86	40	1464.0	184.08
50	442.55	17.07	50	950.25	78.26	50	1472.9	186.29
9° 00′	450.93	17.72	19° 00′	958.81	79.67	29° 00′	1481.8	188.51
10	459.32	18.38	10	967.38	81.09	10	1490.7	190.74
20	467.71	19.06	20	975.96	82.53	20	1499.6	192.99
30	476.10	19.75	30	984.53	83.97	30	1508.5	195.25
40	484.49	20.45	40	993.12	85.43	40	1517.4	197.53
50	492.88	21.16	50	1001.7	86.90	50	1526.3	199.82
10° 00′	501.28	21.89	20° 00′	1010.3	88.39	30° 00′	1535.3	202.12
10	509.68	22.62	10	1018.9	89.89	10	1544.2	204.44
20	518.08	23.38	20	1027.5	91.40	20	1553.1	206.77
30	526.48	24.14	30	1036.1	92.92	30	1562.1	209.12
40	534.89	24.91	40	1044.7	94.46	40	1571.0	211.48
50	543.29	25.70	50	1053.3	96.01	50	1580.0	213.86

TABLE IX TANGENTS AND EXTERNALS TO A 1° CURVE*
—Chord Definition *(Continued)*

Angle	Tangent	External	Angle	Tangent	External	Angle	Tangent	External
31° 00′	1589.0	216.3	41° 00′	2142.2	387.4	51° 00′	2732.9	618.4
10	1598.0	218.7	10	2151.7	390.7	10	2743.1	622.8
20	1606.9	221.1	20	2161.2	394.1	20	2753.4	627.2
30	1615.9	223.5	30	2170.8	397.4	30	2763.7	631.7
40	1624.9	226.0	40	2180.3	400.8	40	2773.9	636.2
50	1633.9	228.4	50	2189.9	404.2	50	2784.2	640.7
32° 00′	1643.0	230.9	42° 00′	2199.4	407.6	52° 00′	2794.5	645.2
10	1652.0	233.4	10	2209.0	411.1	10	2804.9	649.7
20	1661.0	235.9	20	2218.6	414.5	20	2815.2	654.3
30	1670.0	238.4	30	2228.1	418.0	30	2825.6	658.8
40	1679.1	241.0	40	2237.7	421.4	40	2835.9	663.4
50	1688.1	243.5	50	2247.3	425.0	50	2846.3	668.0
33° 00′	1697.2	246.1	43° 00′	2257.0	428.5	53° 00′	2856.7	672.7
10	1706.3	248.7	10	2266.6	432.0	10	2867.1	677.3
20	1715.3	251.3	20	2276.2	435.6	20	2877.5	682.0
30	1724.4	253.9	30	2285.9	439.2	30	2888.0	686.7
40	1733.5	256.5	40	2295.6	442.8	40	2898.4	691.4
50	1742.6	259.1	50	2305.2	446.4	50	2908.9	696.1
34° 00′	1751.7	261.8	44° 00′	2314.9	450.0	54° 00′	2929.4	700.9
10	1760.8	264.5	10	2324.6	453.6	10	2929.9	707.7
20	1770.0	267.2	20	2334.3	457.3	20	2940.4	710.5
30	1779.1	269.9	30	2344.1	461.0	30	2951.0	715.3
40	1788.2	272.6	40	2353.8	464.6	40	2961.5	720.1
50	1797.4	275.3	50	2363.5	468.4	50	2972.1	725.0
35° 00′	1806.6	278.1	45° 00	2373.3	472.1	55° 00	2982.7	729.9
10	1815.7	280.8	10	2383.1	475.8	10	2993.3	734.8
20	1824.9	283.6	20	2392.8	479.6	20	3003.9	739.7
30	1834.1	286.4	30	2402.6	483.4	30	3014.5	744.6
40	1843.3	289.2	40	2412.4	487.2	40	3025.2	749.6
50	1852.5	292.0	50	2422.3	491.0	50	3035.8	754.6
36° 00	1861.7	294.9	46° 00	2432.1	494.8	56° 00	3046.5	759.6
10	1870.9	297.7	10	2441.9	498.7	10	3057.2	764.6
20	1880.1	300.6	20	2451.8	502.5	20	3067.9	769.7
30	1889.4	303.5	30	2461.7	506.4	30	3078.7	774.7
40	1898.6	306.4	40	2471.5	510.3	40	3089.4	779.8
50	1907.9	309.3	50	2481.4	514.3	50	3100.2	784.9
37° 00	1917.1	312.2	47° 00	2491.3	518.2	57° 00	3110.9	790.1
10	1926.4	315.2	10	2501.2	522.2	10	3121.7	795.2
20	1935.7	318.1	20	2511.2	526.1	20	3132.6	800.4
30	1945.0	321.1	30	2521.1	530.1	30	3143.4	805.6
40	1954.3	324.1	40	2531.1	434.2	40	3154.2	810.9
50	1963.6	327.1	50	2541.0	538.2	50	3165.1	816.1
38° 00	1972.9	330.2	48° 00	2551.0	542.2	58° 00	3176.0	821.4
10	1982.2	333.2	10	2561.0	546.3	10	3186.9	826.7
20	1991.5	336.3	20	2571.0	550.4	20	3197.8	832.0
30	2000.9	339.3	30	2581.0	554.5	30	3208.8	837.3
40	2010.2	342.4	40	2591.0	558.6	40	3219.7	842.7
50	2019.6	345.5	50	2601.1	562.8	50	3230.7	848.1
39° 00	2029.0	348.6	49° 00	2611.2	566.9	59° 00	3241.7	853.5
10	2038.4	351.8	10	2621.2	571.1	10	3252.7	858.9
20	2047.8	354.9	20	2631.3	575.3	20	3263.7	864.3
30	2057.2	358.1	30	2641.4	579.5	30	3274.8	869.8
40	2066.6	361.3	40	2651.5	583.8	40	3285.8	875.3
50	2076.0	364.5	50	2661.6	588.0	50	3296.9	880.8
40° 00	2085.4	367.7	50° 00′	2671.8	592.3	60° 00′	3308.0	886.4
10	2094.9	371.0	10	2681.9	596.9	10	3319.1	892.0
20	2104.3	374.2	20	2692.1	600.9	20	3330.3	897.5
30	2113.8	377.5	30	2702.3	605.3	30	3341.4	903.2
40	2123.3	380.8	40	2712.5	609.6	40	3352.6	908.8
50	2132.7	384.1	50	2722.7	614.0	50	3363.8	914.5

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TABLE X LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
100	00000	00043	00087	00130	00173	00217	00260	00303	00346	00389
1	0432	0475	0518	0561	0604	0647	0689	0732	0775	0817
2	0860	0903	0945	0988	1030	1072	1115	1157	1199	1242
3	1284	1326	1368	1410	1452	1494	1536	1578	1620	1662
4	1703	1745	1787	1828	1870	1912	1953	1995	2036	2078
5	2119	2160	2202	2243	2284	2325	2366	2407	2449	2490
6	2531	2572	2612	2653	2694	2735	2776	2816	2857	2898
7	2938	2979	3019	3060	3100	3141	3181	3222	3262	3302
8	3342	3383	3423	3463	3503	3543	3583	3623	3663	3703
9	3743	3782	3822	3862	3902	3941	3981	4021	4060	4100
110	04139	04179	04218	04258	04297	04336	04376	04415	04454	04493
1	4532	4571	4610	4650	4689	4727	4766	4805	4844	4883
2	4922	4961	4999	5038	5077	5115	5154	5192	5231	5269
3	5308	5346	5385	5423	5461	5500	5538	5576	5614	5652
4	5690	5729	5767	5805	5843	5881	5918	5956	5994	6032
5	6070	6108	6145	6183	6221	6258	6296	6333	6371	6408
6	6446	6483	6521	6558	6595	6633	6670	6707	6744	6781
7	6819	6856	6893	6930	6967	7004	7041	7078	7115	7151
8	7188	7225	7262	7298	7335	7372	7408	7445	7482	7518
9	7555	7591	7628	7664	7700	7737	7773	7809	7846	7882
120	07918	07954	07990	08027	08063	08099	08135	08171	08207	08243
1	8279	8314	8350	8386	8422	8458	8493	8529	8565	8600
2	8636	8672	8707	8743	8778	8814	8849	8884	8920	8955
3	8991	9026	9061	9096	9132	9167	9202	9237	9272	9307
4	9342	9377	9412	9447	9482	9517	9552	9587	9621	9656
5	9691	9726	9760	9795	9830	9864	9899	9934	9968	10003
6	10037	10072	10106	10140	10175	10209	10243	10278	10312	0346
7	0380	0415	0449	0483	0517	0551	0585	0619	0653	0687
8	0721	0755	0789	0823	0857	0890	0924	0958	0992	1025
9	1059	1093	1126	1160	1193	1227	1261	1294	1327	1361
130	11394	11428	11461	11494	11528	11561	11594	11628	11661	11694
1	1727	1760	1793	1826	1860	1893	1926	1959	1992	2024
2	2057	2090	2123	2156	2189	2222	2254	2287	2320	2352
3	2385	2418	2450	2483	2516	2548	2581	2613	2646	2678
4	2710	2743	2775	2808	2840	2872	2905	2937	2969	3001
5	3033	3066	3098	3130	3162	3194	3226	3258	3290	3322
6	3354	3386	3418	3450	3481	3513	3545	3577	3609	3640
7	3672	3704	3735	3767	3799	3830	3862	3893	3925	3956
8	3988	4019	4051	4082	4114	4145	4176	4208	4239	4270
9	4301	4333	4364	4395	4426	4457	4489	4520	4551	4582
140	14613	14644	14675	14706	14737	14768	14799	14829	14860	14891
1	4922	4953	4983	5014	5045	5076	5106	5137	5168	5198
2	5229	5259	5290	5320	5351	5381	5412	5442	5473	5503
3	5534	5564	5594	5625	5655	5685	5715	5746	5776	5806
4	5836	5866	5897	5927	5957	5987	6017	6047	6077	6107
5	6137	6167	6197	6227	6256	6286	6316	6346	6376	6406
6	6435	6465	6495	6524	6554	6584	6613	6643	6673	6702
7	6732	6761	6791	6820	6850	6879	6909	6938	6967	6997
8	7026	7056	7085	7114	7143	7173	7202	7231	7260	7289
9	7319	7348	7377	7406	7435	7464	7493	7522	7551	7580
150	17609	17638	17667	17696	17725	17754	17782	17811	17840	17869

N	0	1	2	3	4	5	6	7	8	9
150	17609	17638	17667	17696	17725	17754	17782	17811	17840	17869
1	7898	7926	7955	7984	8013	8041	8070	8099	8127	8156
2	8184	8213	8241	8270	8298	8327	8355	8384	8412	8441
3	8469	8498	8526	8554	8583	8611	8639	8667	8696	8724
4	8752	8780	8808	8837	8865	8893	8921	8949	8977	9005
5	9033	9061	9089	9117	9145	9173	9201	9229	9257	9285
6	9312	9340	9368	9396	9424	9451	9479	9507	9535	9562
7	9590	9618	9645	9673	9700	9728	9756	9783	9811	9838
8	9866	9893	9921	9948	9976	20003	20030	20058	20085	20112
9	20140	20167	20194	20222	20249	0276	0303	0330	0358	0385
160	20412	20439	20466	20493	20520	20548	20575	20602	20629	20656
1	0683	0710	0737	0763	0790	0817	0844	0871	0898	0925
2	0952	0978	1005	1032	1059	1085	1112	1139	1165	1192
3	1219	1245	1272	1299	1325	1352	1378	1405	1431	1458
4	1484	1511	1537	1564	1590	1617	1643	1669	1696	1722
5	1748	1775	1801	1827	1854	1880	1906	1932	1958	1985
6	2011	2037	2063	2089	2115	2141	2167	2194	2220	2246
7	2272	2298	2324	2350	2376	2401	2427	2453	2479	2505
8	2531	2557	2583	2608	2634	2660	2686	2712	2737	2763
9	2789	2814	2840	2866	2891	2917	2943	2968	2994	3019
170	23045	23070	23096	23121	23147	23172	23198	23223	23249	23274
1	3300	3325	3350	3376	3401	3426	3452	3477	3502	3528
2	3553	3578	3603	3629	3654	3679	3704	3729	3754	3779
3	3805	3830	3855	3880	3905	3930	3955	3980	4005	4030
4	4055	4080	4105	4130	4155	4180	4204	4229	4254	4279
5	4304	4329	4353	4378	4403	4428	4452	4477	4502	4527
6	4551	4576	4601	4625	4650	4674	4699	4724	4748	4773
7	4797	4822	4846	4871	4895	4920	4944	4969	4993	5018
8	5042	5066	5091	5115	5139	5164	5188	5212	5237	5261
9	5285	5310	5334	5358	5382	5406	5431	5455	5479	5503
180	25527	25551	25575	25600	25624	25648	25672	25696	25720	25744
1	5768	5792	5816	5840	5864	5888	5912	5935	5959	5983
2	6007	6031	6055	6079	6102	6126	6150	6174	6198	6221
3	6245	6269	6293	6316	6340	6364	6387	6411	6435	6458
4	6482	6505	6529	6553	6576	6600	6623	6647	6670	6694
5	6717	6741	6764	6788	6811	6834	6858	6881	6905	6928
6	6951	6975	6998	7021	7045	7068	7091	7114	7138	7161
7	7184	7207	7231	7254	7277	7300	7323	7346	7370	7393
8	7416	7439	7462	7485	7508	7531	7554	7577	7600	7623
9	7646	7669	7692	7715	7738	7761	7784	7807	7830	7852
190	27875	27898	27921	27944	27967	27989	28012	28035	28058	28081
1	8103	8126	8149	8171	8194	8217	8240	8262	8285	8307
2	8330	8353	8375	8398	8421	8443	8466	8488	8511	8533
3	8556	8578	8601	8623	8646	8668	8691	8713	8735	8758
4	8780	8803	8825	8847	8870	8892	8914	8937	8959	8981
5	9003	9026	9048	9070	9092	9115	9137	9159	9181	9203
6	9226	9248	9270	9292	9314	9336	9358	9380	9403	9425
7	9447	9469	9491	9513	9535	9557	9579	9601	9623	9645
8	9667	9688	9710	9732	9754	9776	9798	9820	9842	9863
9	9885	9907	9929	9951	9973	9994	30016	30038	30060	30081
200	30103	30125	30146	30168	30190	30211	30233	30255	30276	30298

N	0	1	2	3	4	5	6	7	8	9
200	30103	30125	30146	30168	30190	30211	30233	30255	30276	30298
1	0320	0341	0363	0384	0406	0428	0449	0471	0492	0514
2	0535	0557	0578	0600	0621	0643	0664	0685	0707	0728
3	0750	0771	0792	0814	0835	0856	0878	0899	0920	0942
4	0963	0984	1006	1027	1048	1069	1091	1112	1133	1154
5	1175	1197	1218	1239	1260	1281	1302	1323	1345	1366
6	1387	1408	1429	1450	1471	1492	1513	1534	1555	1576
7	1597	1618	1639	1660	1681	1702	1723	1744	1765	1785
8	1806	1827	1848	1869	1890	1911	1931	1952	1973	1994
9	2015	2035	2056	2077	2098	2118	2139	2160	2181	2201
210	32222	32243	32263	32284	32305	32325	32346	32366	32387	32408
1	2428	2449	2469	2490	2510	2531	2552	2572	2593	2613
2	2634	2654	2675	2695	2715	2736	2756	2777	2797	2818
3	2838	2858	2879	2899	2919	2940	2960	2980	3001	3021
4	3041	3062	3082	3102	3122	3143	3163	3183	3203	3224
5	3244	3264	3284	3304	3325	3345	3365	3385	3405	3425
6	3445	3465	3486	3506	3526	3546	3566	3586	3606	3626
7	3646	3666	3686	3706	3726	3746	3766	3786	3806	3826
8	3846	3866	3885	3905	3925	3945	3965	3985	4005	4025
9	4044	4064	4084	4104	4124	4143	4163	4183	4203	4223
220	34242	34262	34282	34301	34321	34341	34361	34380	34400	34420
1	4439	4459	4479	4498	4518	4537	4557	4577	4596	4616
2	4635	4655	4674	4694	4713	4733	4753	4772	4792	4811
3	4830	4850	4869	4889	4908	4928	4947	4967	4986	5005
4	5025	5044	5064	5083	5102	5122	5141	5160	5180	5199
5	5218	5238	5257	5276	5295	5315	5334	5353	5372	5392
6	5411	5430	5449	5468	5488	5507	5526	5545	5564	5583
7	5603	5622	5641	5660	5679	5698	5717	5736	5755	5774
8	5793	5813	5832	5851	5870	5889	5908	5927	5946	5965
9	5984	6003	6021	6040	6059	6078	6097	6116	6135	6154
230	36173	36192	36211	36229	36248	36267	36286	36305	36324	36342
1	6361	6380	6399	6418	6436	6455	6474	6493	6511	6530
2	6549	6568	6586	6605	6624	6642	6661	6680	6698	6717
3	6736	6754	6773	6791	6810	6829	6847	6866	6884	6903
4	6922	6940	6959	6977	6996	7014	7033	7051	7070	7088
5	7107	7125	7144	7162	7181	7199	7218	7236	7254	7273
6	7291	7310	7328	7346	7365	7383	7401	7420	7438	7457
7	7475	7493	7511	7530	7548	7566	7585	7603	7621	7639
8	7658	7676	7694	7712	7731	7749	7767	7785	7803	7822
9	7840	7858	7876	7894	7912	7931	7949	7967	7985	8003
240	38021	38039	38057	38075	38093	38112	38130	38148	38166	38184
1	8202	8220	8238	8256	8274	8292	8310	8328	8346	8364
2	8382	8399	8417	8435	8453	8471	8489	8507	8525	8543
3	8561	8578	8596	8614	8632	8650	8668	8686	8703	8721
4	8739	8757	8775	8792	8810	8828	8846	8863	8881	8899
5	8917	8934	8952	8970	8987	9005	9023	9041	9058	9076
6	9094	9111	9129	9146	9164	9182	9199	9217	9235	9252
7	9270	9287	9305	9322	9340	9358	9375	9393	9410	9428
8	9445	9463	9480	9498	9515	9533	9550	9568	9585	9602
9	9620	9637	9655	9672	9690	9707	9724	9742	9759	9777
250	39794	39811	39829	39846	39863	39881	39898	39915	39933	39950

TABLE X LOGARITHMS OF NUMBERS *(Continued)*

N	0	1	2	3	4	5	6	7	8	9
250	39794	39811	39829	39846	39863	39881	39898	39915	39933	39950
1	9967	9985	40002	40019	40037	40054	40071	40088	40106	40123
2	40140	40157	0175	0192	0209	0226	0243	0261	0278	0295
3	0312	0329	0346	0364	0381	0398	0415	0432	0449	0466
4	0483	0500	0518	0535	0552	0569	0586	0603	0620	0637
5	0654	0671	0688	0705	0722	0739	0756	0773	0790	0807
6	0824	0841	0858	0875	0892	0909	0926	0943	0960	0976
7	0993	1010	1027	1044	1061	1078	1095	1111	1128	1145
8	1162	1179	1196	1212	1229	1246	1263	1280	1296	1313
9	1330	1347	1363	1380	1397	1414	1430	1447	1464	1481
260	41497	41514	41531	41547	41564	41581	41597	41614	41631	41647
1	1664	1681	1697	1714	1731	1747	1764	1780	1797	1814
2	1830	1847	1863	1880	1896	1913	1929	1946	1963	1979
3	1996	2012	2029	2045	2062	2078	2095	2111	2127	2144
4	2160	2177	2193	2210	2226	2243	2259	2275	2292	2308
5	2325	2341	2357	2374	2390	2406	2423	2439	2455	2472
6	2488	2504	2521	2537	2553	2570	2586	2602	2619	2635
7	2651	2667	2684	2700	2716	2732	2749	2765	2781	2797
8	2813	2830	2846	2862	2878	2894	2911	2927	2943	2959
9	2975	2991	3008	3024	3040	3056	3072	3088	3104	3120
270	43136	43152	43169	43185	43201	43217	43233	43249	43265	43281
1	3297	3313	3329	3345	3361	3377	3393	3409	3425	3441
2	3457	3473	3489	3505	3521	3537	3553	3569	3584	3600
3	3616	3632	3648	3664	3680	3696	3712	3727	3743	3759
4	3775	3791	3807	3823	3838	3854	3870	3886	3902	3917
5	3933	3949	3965	3981	3996	4012	4028	4044	4059	4075
6	4091	4107	4122	4138	4154	4170	4185	4201	4217	4232
7	4248	4264	4279	4295	4311	4326	4342	4358	4373	4389
8	4404	4420	4436	4451	4467	4483	4498	4514	4529	4545
9	4560	4576	4592	4607	4623	4638	4654	4669	4685	4700
280	44716	44731	44747	44762	44778	44793	44809	44824	44840	44855
1	4871	4886	4902	4917	4932	4948	4963	4979	4994	5010
2	5025	5040	5056	5071	5086	5102	5117	5133	5148	5163
3	5179	5194	5209	5225	5240	5255	5271	5286	5301	5317
4	5332	5347	5362	5378	5393	5408	5423	5439	5454	5469
5	5484	5500	5515	5530	5545	5561	5576	5591	5606	5621
6	5637	5652	5667	5682	5697	5712	5728	5743	5758	5773
7	5788	5803	5818	5834	5849	5864	5879	5894	5909	5924
8	5939	5954	5969	5984	6000	6015	6030	6045	6060	6075
9	6090	6105	6120	6135	6150	6165	6180	6195	6210	6225
290	46240	46255	46270	46285	46300	46315	46330	46345	46359	46374
1	6389	6404	6419	6434	6449	6464	6479	6494	6509	6523
2	6538	6553	6568	6583	6598	6613	6627	6642	6657	6672
3	6687	6702	6716	6731	6746	6761	6776	6790	6805	6820
4	6835	6850	6864	6879	6894	6909	6923	6938	6953	6967
5	6982	6997	7012	7026	7041	7056	7070	7085	7100	7114
6	7129	7144	7159	7173	7188	7202	7217	7232	7246	7261
7	7276	7290	7305	7319	7334	7349	7363	7378	7392	7407
8	7422	7436	7451	7465	7480	7494	7509	7524	7538	7553
9	7567	7582	7596	7611	7625	7640	7654	7669	7683	7698
300	47712	47727	47741	47756	47770	47784	47799	47813	47828	47842

TABLE X LOGARITHMS OF NUMBERS *(Continued)*

N	0	1	2	3	4	5	6	7	8	9
300	47712	47727	47741	47756	47770	47784	47799	47813	47828	47842
1	7857	7871	7885	7900	7914	7929	7943	7958	7972	7986
2	8001	8015	8029	8044	8058	8073	8087	8101	8116	8130
3	8144	8159	8173	8187	8202	8216	8230	8244	8259	8273
4	8287	8302	8316	8330	8344	8359	8373	8387	8401	8416
5	8430	8444	8458	8473	8487	8501	8515	8530	8544	8558
6	8572	8586	8601	8615	8629	8643	8657	8671	8686	8700
7	8714	8728	8742	8756	8770	8785	8799	8813	8827	8841
8	8855	8869	8883	8897	8911	8926	8940	8954	8968	8982
9	8996	9010	9024	9038	9052	9066	9080	9094	9108	9122
310	49136	49150	49164	49178	49192	49206	49220	49234	49248	49262
1	9276	9290	9304	9318	9332	9346	9360	9374	9388	9402
2	9415	9429	9443	9457	9471	9485	9499	9513	9527	9541
3	9554	9568	9582	9596	9610	9624	9638	9651	9665	9679
4	9693	9707	9721	9734	9748	9762	9776	9790	9803	9817
5	9831	9845	9859	9872	9886	9900	9914	9927	9941	9955
6	9969	9982	9996	50010	50024	50037	50051	50065	50079	50092
7	50106	50120	50133	0147	0161	0174	0188	0202	0215	0229
8	0243	0256	0270	0284	0297	0311	0325	0338	0352	0365
9	0379	0393	0406	0420	0433	0447	0461	0474	0488	0501
320	50515	50529	50542	50556	50569	50583	50596	50610	50623	50637
1	0651	0664	0678	0691	0705	0718	0732	0745	0759	0772
2	0786	0799	0813	0826	0840	0853	0866	0880	0893	0907
3	0920	0934	0947	0961	0974	0987	1001	1014	1028	1041
4	1055	1068	1081	1095	1108	1121	1135	1148	1162	1175
5	1188	1202	1215	1228	1242	1255	1268	1282	1295	1308
6	1322	1335	1348	1362	1375	1388	1402	1415	1428	1441
7	1455	1468	1481	1495	1508	1521	1534	1548	1561	1574
8	1587	1601	1614	1627	1640	1654	1667	1680	1693	1706
9	1720	1733	1746	1759	1772	1786	1799	1812	1825	1838
330	51851	51865	51878	51891	51904	51917	51930	51943	51957	51970
1	1983	1996	2009	2022	2035	2048	2061	2075	2088	2101
2	2114	2127	2140	2153	2166	2179	2192	2205	2218	2231
3	2244	2257	2270	2284	2297	2310	2323	2336	2349	2362
4	2375	2388	2401	2414	2427	2440	2453	2466	2479	2492
5	2504	2517	2530	2543	2556	2569	2582	2595	2608	2621
6	2634	2647	2660	2673	2686	2699	2711	2724	2737	2750
7	2763	2776	2789	2802	2815	2827	2840	2853	2866	2879
8	2892	2905	2917	2930	2943	2956	2969	2982	2994	3007
9	3020	3033	3046	3058	3071	3084	3097	3110	3122	3135
340	53148	53161	53173	53186	53199	53212	53224	53237	53250	53263
1	3275	3288	3301	3314	3326	3339	3352	3364	3377	3390
2	3403	3415	3428	3441	3453	3466	3479	3491	3504	3517
3	3529	3542	3555	3567	3580	3593	3605	3618	3631	3643
4	3656	3668	3681	3694	3706	3719	3732	3744	3757	3769
5	3782	3794	3807	3820	3832	3845	3857	3870	3882	3895
6	3908	3920	3933	3945	3958	3970	3983	3995	4008	4020
7	4033	4045	4058	4070	4083	4095	4108	4120	4133	4145
8	4158	4170	4183	4195	4208	4220	4233	4245	4258	4270
9	4283	4295	4307	4320	4332	4345	4357	4370	4382	4394
350	54407	54419	54432	54444	54456	54469	54481	54494	54506	54518

N	0	1	2	3	4	5	6	7	8	9
350	54407	54419	54432	54444	54456	54469	54481	54494	54506	54518
1	4531	4543	4555	4568	4580	4593	4605	4617	4630	4642
2	4654	4667	4679	4691	4704	4716	4728	4741	4753	4765
3	4777	4790	4802	4814	4827	4839	4851	4864	4876	4888
4	4900	4913	4925	4937	4949	4962	4974	4986	4998	5011
5	5023	5035	5047	5060	5072	5084	5096	5108	5121	5133
6	5145	5157	5169	5182	5194	5206	5218	5230	5242	5255
7	5267	5279	5291	5303	5315	5328	5340	5352	5364	5376
8	5388	5400	5413	5425	5437	5449	5461	5473	5485	5497
9	5509	5522	5534	5546	5558	5570	5582	5594	5606	5618
360	55630	55642	55654	55666	55678	55691	55703	55715	55727	55739
1	5751	5763	5775	5787	5799	5811	5823	5835	5847	5859
2	5871	5883	5895	5907	5919	5931	5943	5955	5967	5979
3	5991	6003	6015	6027	6038	6050	6062	6074	6086	6098
4	6110	6122	6134	6146	6158	6170	6182	6194	6205	6217
5	6229	6241	6253	6265	6277	6289	6301	6312	6324	6336
6	6348	6360	6372	6384	6396	6407	6419	6431	6443	6455
7	6467	6478	6490	6502	6514	6526	6538	6549	6561	6573
8	6585	6597	6608	6620	6632	6644	6656	6667	6679	6691
9	6703	6714	6726	6738	6750	6761	6773	6785	6797	6808
370	56820	56832	56844	56855	56867	56879	56891	56902	65914	56926
1	6937	6949	6961	6972	6984	6996	7008	7019	7031	7043
2	7054	7066	7078	7089	7101	7113	7124	7136	7148	7159
3	7171	7183	7194	7206	7217	7229	7241	7252	7264	7276
4	7287	7299	7310	7322	7334	7345	7357	7368	7380	7392
5	7403	7415	7426	7438	7449	7461	7473	7484	7496	7507
6	7519	7530	7542	7553	7565	7576	7588	7600	7611	7623
7	7634	7646	7657	7669	7680	7692	7703	7715	7726	7738
8	7749	7761	7772	7784	7795	7807	7818	7830	7841	7852
9	7864	7875	7887	7898	7910	7921	7933	7944	7955	7967
380	57978	57990	58001	58013	58024	58035	58047	58058	58070	58081
1	8092	8104	8115	8127	8138	8149	8161	8172	8184	8195
2	8206	8218	8229	8240	8252	8263	8274	8286	8297	8309
3	8320	8331	8343	8354	8365	8377	8388	8399	8410	8422
4	8433	8444	8456	8467	8478	8490	8501	8512	8524	8535
5	8546	8557	8569	8580	8591	8602	8614	8625	8636	8647
6	8659	8670	8681	8692	8704	8715	8726	8737	8749	8760
7	8771	8782	8794	8805	8816	8827	8838	8850	8861	8872
8	8883	8894	8906	8917	8928	8939	8950	8961	8973	8984
9	8995	9006	9017	9028	9040	9051	9062	9073	9084	9095
390	59106	59118	59129	59140	59151	59162	59173	59184	59195	59207
1	9218	9229	9240	9251	9262	9273	9284	9295	9306	9318
2	9329	9340	9351	9362	9373	9384	9395	9406	9417	9428
3	9439	9450	9461	9472	9483	9494	9506	9517	9528	9539
4	9550	9561	9572	9583	9594	9605	9616	9627	9638	9649
5	9660	9671	9682	9693	9704	9715	9726	9737	9748	9759
6	9770	9780	9791	9802	9813	9824	9835	9846	9857	9868
7	9879	9890	9901	9912	9923	9934	9945	9956	9966	9977
8	9988	9999	60010	60021	60032	60043	60054	60065	60076	60086
9	60097	60108	0119	0130	0141	0152	0163	0173	0184	0195
400	60206	60217	60228	60239	60249	60260	60271	60282	60293	60304

TABLE X LOGARITHMS OF NUMBERS *(Continued)*

N	0	1	2	3	4	5	6	7	8	9
400	60206	60217	60228	60239	60249	60260	60271	60282	60293	60304
1	0314	0325	0336	0347	0358	0369	0379	0390	0401	0412
2	0423	0433	0444	0455	0466	0477	0487	0498	0509	0520
3	0531	0541	0552	0563	0574	0584	0595	0606	0617	0627
4	0638	0649	0660	0670	0681	0692	0703	0713	0724	0735
5	0746	0756	0767	0778	0788	0799	0810	0821	0831	0842
6	0853	0863	0874	0885	0895	0906	0917	0927	0938	0949
7	0959	0970	0981	0991	1002	1013	1023	1034	1045	1055
8	1066	1077	1087	1098	1109	1119	1130	1140	1151	1162
9	1172	1183	1194	1204	1215	1225	1236	1247	1257	1268
410	61278	61289	61300	61310	61321	61331	61342	61352	61363	61374
1	1384	1395	1405	1416	1426	1437	1448	1458	1469	1479
2	1490	1500	1511	1521	1532	1542	1553	1563	1574	1584
3	1595	1606	1616	1627	1637	1648	1658	1669	1679	1690
4	1700	1711	1721	1731	1742	1752	1763	1773	1784	1794
5	1805	1815	1826	1836	1847	1857	1868	1878	1888	1899
6	1909	1920	1930	1941	1951	1962	1972	1982	1993	2003
7	2014	2024	2034	2045	2055	2066	2076	2086	2097	2107
8	2118	2128	2138	2149	2159	2170	2180	2190	2201	2211
9	2221	2232	2242	2252	2263	2273	2284	2294	2304	2315
420	62325	62335	62346	62356	62366	62377	62387	62397	62408	62418
1	2428	2439	2449	2459	2469	2480	2490	2500	2511	2521
2	2531	2542	2552	2562	2572	2583	2593	2603	2613	2624
3	2634	2644	2655	2665	2675	2685	2696	2706	2716	2726
4	2737	2747	2757	2767	2778	2788	2798	2808	2818	2829
5	2839	2849	2859	2870	2880	2890	2900	2910	2921	2931
6	2941	2951	2961	2972	2982	2992	3002	3012	3022	3033
7	3043	3053	3063	3073	3083	3094	3104	3114	3124	3134
8	3144	3155	3165	3175	3185	3195	3205	3215	3225	3236
9	3246	3256	3266	3276	3286	3296	3306	3317	3327	3337
430	63347	63357	63367	63377	63387	63397	63407	63417	63428	63438
1	3448	3458	3468	3478	3488	3498	3508	3518	3528	3538
2	3548	3558	3568	3579	3589	3599	3609	3619	3629	3639
3	3649	3659	3669	3679	3689	3699	3709	3719	3729	3739
4	3749	3759	3769	3779	3789	3799	3809	3819	3829	3839
5	3849	3859	3869	3879	3889	3899	3909	3919	3929	3939
6	3949	3959	3969	3979	3988	3998	4008	4018	4028	4038
7	4048	4058	4068	4078	4088	4098	4108	4118	4128	4137
8	4147	4157	4167	4177	4187	4197	4207	4217	4227	4237
9	4246	4256	4266	4276	4286	4296	4306	4316	4326	4335
440	64345	64355	64365	64375	64385	64395	64404	64414	64424	64434
1	4444	4454	4464	4473	4483	4493	4503	4513	4523	4532
2	4542	4552	4562	4572	4582	4591	4601	4611	4621	4631
3	4640	4650	4660	4670	4680	4689	4699	4709	4719	4729
4	4738	4748	4758	4768	4777	4787	4797	4807	4816	4826
5	4836	4846	4856	4865	4875	4885	4895	4904	4914	4924
6	4933	4943	4953	4963	4972	4982	4992	5002	5011	5021
7	5031	5040	5050	5060	5070	5079	5089	5099	5108	5118
8	5128	5137	5147	5157	5167	5176	5186	5196	5205	5215
9	5225	5234	5244	5254	5263	5273	5283	5292	5302	5312
450	65321	65331	65341	65350	65360	65369	65379	65389	65398	65408

N	0	1	2	3	4	5	6	7	8	9
450	65321	65331	65341	65350	65360	65369	65379	65389	65398	65408
1	5418	5427	5437	5447	5456	5466	5475	5485	5495	5504
2	5514	5523	5533	5543	5552	5562	5571	5581	5591	5600
3	5610	5619	5629	5639	5648	5658	5667	5677	5686	5696
4	5706	5715	5725	5734	5744	5753	5763	5772	5782	5792
5	5801	5811	5820	5830	5839	5849	5858	5868	5877	5887
6	5896	5906	5916	5925	5935	5944	5954	5963	5973	5982
7	5992	6001	6011	6020	6030	6039	6049	6058	6068	6077
8	6087	6096	6106	6115	6124	6134	6143	6153	6162	6172
9	6181	6191	6200	6210	6219	6229	6238	6247	6257	6266
460	66276	66285	66295	66304	66314	66323	66332	66342	66351	66361
1	6370	6380	6389	6398	6408	6417	6427	6436	6445	6455
2	6464	6474	6483	6492	6502	6511	6521	6530	6539	6549
3	6558	6567	6577	6586	6596	6605	6614	6624	6633	6642
4	6652	6661	6671	6680	6689	6699	6708	6717	6727	6736
5	6745	6755	6764	6773	6783	6792	6801	6811	6820	6829
6	6839	6848	6857	6867	6876	6885	6894	6904	6913	6922
7	6932	6941	6950	6960	6969	6978	6987	6997	7006	7015
8	7025	7034	7043	7052	7062	7071	7080	7089	7099	7108
9	7117	7127	7136	7145	7154	7164	7173	7182	7191	7201
470	67210	67219	67228	67237	67247	67256	67265	67274	67284	67293
1	7302	7311	7321	7330	7339	7348	7357	7367	7376	7385
2	7394	7403	7413	7422	7431	7440	7449	7459	7468	7477
3	7486	7495	7504	7514	7523	7532	7541	7550	7560	7569
4	7578	7587	7596	7605	7614	7624	7633	7642	7651	7660
5	7669	7679	7688	7697	7706	7715	7724	7733	7742	7752
6	7761	7770	7779	7788	7797	7806	7815	7825	7834	7843
7	7852	7861	7870	7879	7888	7897	7906	7916	7925	7934
8	7943	7952	7961	7970	7979	7988	7997	8006	8015	8024
9	8034	8043	8052	8061	8070	8079	8088	8097	8106	8115
480	68124	68133	68142	68151	68160	68169	68178	68187	68196	68205
1	8215	8224	8233	8242	8251	8260	8269	8278	8287	8296
2	8305	8314	8323	8332	8341	8350	8359	8368	8377	8386
3	8395	8404	8413	8422	8431	8440	8449	8458	8467	8476
4	8485	8494	8502	8511	8520	8529	8538	8547	8556	8565
5	8574	8583	8592	8601	8610	8619	8628	8637	8646	8655
6	8664	8673	8681	8690	8699	8708	8717	8726	8735	8744
7	8753	8762	8771	8780	8789	8797	8806	8815	8824	8833
8	8842	8851	8860	8869	8878	8886	8895	8904	8913	8922
9	8931	8940	8949	8958	8966	8975	8984	8993	9002	9011
490	69020	69028	69037	69046	69055	69064	69073	69082	69090	69099
1	9108	9117	9126	9135	9144	9152	9161	9170	9179	9188
2	9197	9205	9214	9223	9232	9241	9249	9258	9267	9276
3	9285	9294	9302	9311	9320	9329	9338	9346	9355	9364
4	9373	9381	9390	9399	9408	9417	9425	9434	9443	9452
5	9461	9469	9478	9487	9496	9504	9513	9522	9531	9539
6	9548	9557	9566	9574	9583	9592	9601	9609	9618	9627
7	9636	9644	9653	9662	9671	9679	9688	9697	9705	9714
8	9723	9732	9740	9749	9758	9767	9775	9784	9793	9801
9	9810	9819	9827	9836	9845	9854	9862	9871	9880	9888
500	69897	69906	69914	69923	69932	69940	69949	69958	69966	69975

N	0	1	2	3	4	5	6	7	8	9
500	69897	69906	69914	69923	69932	69940	69949	69958	69966	69975
1	9984	9992	70001	70010	70018	70027	70036	70044	70053	70062
2	70070	70079	0088	0096	0105	0114	0122	0131	0140	0148
3	0157	0165	0174	0183	0191	0200	0209	0217	0226	0234
4	0243	0252	0260	0269	0278	0286	0295	0303	0312	0321
5	0329	0338	0346	0355	0364	0372	0381	0389	0398	0406
6	0415	0424	0432	0441	0449	0458	0467	0475	0484	0492
7	0501	0509	0518	0526	0535	0544	0552	0561	0569	0578
8	0586	0595	0603	0612	0621	0629	0638	0646	0655	0663
9	0672	0680	0689	0697	0706	0714	0723	0731	0740	0749
510	70757	70766	70774	70783	70791	70800	70808	70817	70825	70834
1	0842	0851	0859	0868	0876	0885	0893	0902	0910	0919
2	0927	0935	0944	0952	0961	0969	0978	0986	0995	1003
3	1012	1020	1029	1037	1046	1054	1063	1071	1079	1088
4	1096	1105	1113	1122	1130	1139	1147	1155	1164	1172
5	1181	1189	1198	1206	1214	1223	1231	1240	1248	1257
6	1265	1273	1282	1290	1299	1307	1315	1324	1332	1341
7	1349	1357	1366	1374	1383	1391	1399	1408	1416	1425
8	1433	1441	1450	1458	1466	1475	1483	1492	1500	1508
9	1517	1525	1533	1542	1550	1559	1567	1575	1584	1592
520	71600	71609	71617	71625	71634	71642	71650	71659	71667	71675
1	1684	1692	1700	1709	1717	1725	1734	1742	1750	1759
2	1767	1775	1784	1792	1800	1809	1817	1825	1834	1842
3	1850	1858	1867	1875	1883	1892	1900	1908	1917	1925
4	1933	1941	1950	1958	1966	1975	1983	1991	1999	2008
5	2016	2024	2032	2041	2049	2057	2066	2074	2082	2090
6	2099	2107	2115	2123	2132	2140	2148	2156	2165	2173
7	2181	2189	2198	2206	2214	2222	2230	2239	2247	2255
8	2263	2272	2280	2288	2296	2304	2313	2321	2329	2337
9	2346	2354	2362	2370	2378	2387	2395	2403	2411	2419
530	72428	72436	72444	72452	72460	72469	72477	72485	72493	72501
1	2509	2518	2526	2534	2542	2550	2558	2567	2575	2583
2	2591	2599	2607	2616	2624	2632	2640	2648	2656	2665
3	2673	2681	2689	2697	2705	2713	2722	2730	2738	2746
4	2754	2762	2770	2779	2787	2795	2803	2811	2819	2827
5	2835	2843	2852	2860	2868	2876	2884	2892	2900	2908
6	2916	2925	2933	2941	2949	2957	2965	2973	2981	2989
7	2997	3006	3014	3022	3030	3038	3046	3054	3062	3070
8	3078	3086	3094	3102	3111	3119	3127	3135	3143	3151
9	3159	3167	3175	3183	3191	3199	3207	3215	3223	3231
540	73239	73247	73255	73263	73272	73280	73288	73296	73304	73312
1	3320	3328	3336	3344	3352	3360	3368	3376	3384	3392
2	3400	3408	3416	3424	3432	3440	3448	3456	3464	3472
3	3480	3488	3496	3504	3512	3520	3528	3536	3544	3552
4	3560	3568	3576	3584	3592	3600	3608	3616	3624	3632
5	3640	3648	3656	3664	3672	3679	3687	3695	3703	3711
6	3719	3727	3735	3743	3751	3759	3767	3775	3783	3791
7	3799	3807	3815	3823	3830	3838	3846	3854	3862	3870
8	3878	3886	3894	3902	3910	3918	3926	3933	3941	3949
9	3957	3965	3973	3981	3989	3997	4005	4013	4020	4028
550	74036	74044	74052	74060	74068	74076	74084	74092	74099	74107

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550	74036	74044	74052	74060	74068	74076	74084	74092	74099	74107
1	4115	4123	4131	4139	4147	4155	4162	4170	4178	4186
2	4194	4202	4210	4218	4225	4233	4241	4249	4257	4265
3	4273	4280	4288	4296	4304	4312	4320	4327	4335	4343
4	4351	4359	4367	4374	4382	4390	4398	4406	4414	4421
5	4429	4437	4445	4453	4461	4468	4476	4484	4492	4500
6	4507	4515	4523	4531	4539	4547	4554	4562	4570	4578
7	4586	4593	4601	4609	4617	4624	4632	4640	4648	4656
8	4663	4671	4679	4687	4695	4702	4710	4718	4726	4733
9	4741	4749	4757	4764	4772	4780	4788	4796	4803	4811
560	74819	74827	74834	74842	74850	74858	74865	74873	74881	74889
1	4896	4904	4912	4920	4927	4935	4943	4950	4958	4966
2	4974	4981	4989	4997	5005	5012	5020	5028	5035	5043
3	5051	5059	5066	5074	5082	5089	5097	5105	5113	5120
4	5128	5136	5143	5151	5159	5166	5174	5182	5189	5197
5	5205	5213	5220	5228	5236	5243	5251	5259	5266	5274
6	5282	5289	5297	5305	5312	5320	5328	5335	5343	5351
7	5358	5366	5374	5381	5389	5397	5404	5412	5420	5427
8	5435	5442	5450	5458	5465	5473	5481	5488	5496	5504
9	5511	5519	5526	5534	5542	5549	5557	5565	5572	5580
570	75587	75595	75603	75610	75618	75626	75633	75641	75648	75656
1	5664	5671	5679	5686	5694	5702	5709	5717	5724	5732
2	5740	5747	5755	5762	5770	5778	5785	5793	5800	5808
3	5815	5823	5831	5838	5846	5853	5861	5868	5876	5884
4	5891	5899	5906	5914	5921	5929	5937	5944	5952	5959
5	5967	5974	5982	5989	5997	6005	6012	6020	6027	6035
6	6042	6050	6057	6065	6072	6080	6087	6095	6103	6110
7	6118	6125	6133	6140	6148	6155	6163	6170	6178	6185
8	6193	6200	6208	6215	6223	6230	6238	6245	6253	6260
9	6268	6275	6283	6290	6298	6305	6313	6320	6328	6335
580	76343	76350	76358	76365	76373	76380	76388	76395	76403	76410
1	6418	6425	6433	6440	6448	6455	6462	6470	6477	6485
2	6492	6500	6507	6515	6522	6530	6537	6545	6552	6559
3	6567	6574	6582	6589	6597	6604	6612	6619	6626	6634
4	6641	6649	6656	6664	6671	6678	6686	6693	6701	6708
5	6716	6723	6730	6738	6745	6753	6760	6768	6775	6782
6	6790	6797	6805	6812	6819	6827	6834	6842	6849	6856
7	6864	6871	6879	6886	6893	6901	6908	6916	6923	6930
8	6938	6945	6953	6960	6967	6975	6982	6989	6997	7004
9	7012	7019	7026	7034	7041	7048	7056	7063	7070	7078
590	77085	77093	77100	77107	77115	77122	77129	77137	77144	77151
1	7159	7166	7173	7181	7188	7195	7203	7210	7217	7225
2	7232	7240	7247	7254	7262	7269	7276	7283	7291	7298
3	7305	7313	7320	7327	7335	7342	7349	7357	7364	7371
4	7379	7386	7393	7401	7408	7415	7422	7430	7437	7444
5	7452	7459	7466	7474	7481	7488	7495	7503	7510	7517
6	7525	7532	7539	7546	7554	7561	7568	7576	7583	7590
7	7597	7605	7612	7619	7627	7634	7641	7648	7656	7663
8	7670	7677	7685	7692	7699	7706	7714	7721	7728	7735
9	7743	7750	7757	7764	7772	7779	7786	7793	7801	7808
600	77815	77822	77830	77837	77844	77851	77859	77866	77873	77880

TABLE X LOGARITHMS OF NUMBERS *(Continued)*

N	0	1	2	3	4	5	6	7	8	9
600	77815	77822	77830	77837	77844	77851	77859	77866	77873	77880
1	7887	7895	7902	7909	7916	7924	7931	7938	7945	7952
2	7960	7967	7974	7981	7988	7996	8003	8010	8017	8025
3	8032	8039	8046	8053	8061	8068	8075	8082	8089	8097
4	8104	8111	8118	8125	8132	8140	8147	8154	8161	8168
5	8176	8183	8190	8197	8204	8211	8219	8226	8233	8240
6	8247	8254	8262	8269	8276	8283	8290	8297	8305	8312
7	8319	8326	8333	8340	8347	8355	8362	8369	8376	8383
8	8390	8398	8405	8412	8419	8426	8433	8440	8447	8455
9	8462	8469	8476	8483	8490	8497	8504	8512	8519	8526
610	78533	78540	78547	78554	78561	78569	78576	78583	78590	78597
1	8604	8611	8618	8625	8633	8640	8647	8654	8661	8668
2	8675	8682	8689	8696	8704	8711	8718	8725	8732	8739
3	8746	8753	8760	8767	8774	8781	8789	8796	8803	8810
4	8817	8824	8831	8838	8845	8852	8859	8866	8873	8880
5	8888	8895	8902	8909	8916	8923	8930	8937	8944	8951
6	8958	8965	8972	8979	8986	8993	9000	9007	9014	9021
7	9029	9036	9043	9050	9057	9064	9071	9078	9085	9092
8	9099	9106	9113	9120	9127	9134	9141	9148	9155	9162
9	9169	9176	9183	9190	9197	9204	9211	9218	9225	9232
620	79239	79246	79253	79260	79267	79274	79281	79288	79295	79302
1	9309	9316	9323	9330	9337	9344	9351	9358	9365	9372
2	9379	9386	9393	9400	9407	9414	9421	9428	9435	9442
3	9449	9456	9463	9470	9477	9484	9491	9498	9505	9511
4	9518	9525	9532	9539	9546	9553	9560	9567	9574	9581
5	9588	9595	9602	9609	9616	9623	9630	9637	9644	9650
6	9657	9664	9671	9678	9685	9692	9699	9706	9713	9720
7	9727	9734	9741	9748	9754	9761	9768	9775	9782	9789
8	9796	9803	9810	9817	9824	9831	9837	9844	9851	9858
9	9865	9872	9879	9886	9893	9900	9906	9913	9920	9927
630	79934	79941	79948	79955	79962	79969	79975	79982	79989	79996
1	80003	80010	80017	80024	80030	80037	80044	80051	80058	80065
2	0072	0079	0085	0092	0099	0106	0113	0120	0127	0134
3	0140	0147	0154	0161	0168	0175	0182	0188	0195	0202
4	0209	0216	0223	0229	0236	0243	0250	0257	0264	0271
5	0277	0284	0291	0298	0305	0312	0318	0325	0332	0339
6	0346	0353	0359	0366	0373	0380	0387	0393	0400	0407
7	0414	0421	0428	0434	0441	0448	0455	0462	0468	0475
8	0482	0489	0496	0502	0509	0516	0523	0530	0536	0543
9	0550	0557	0564	0570	0577	0584	0591	0598	0604	0611
640	80618	80625	80632	80638	80645	80652	80659	80665	80672	80679
1	0686	0693	0699	0706	0713	0720	0726	0733	0740	0747
2	0754	0760	0767	0774	0781	0787	0794	0801	0808	0814
3	0821	0828	0835	0841	0848	0855	0862	0868	0875	0882
4	0889	0895	0902	0909	0916	0922	0929	0936	0943	0949
5	0956	0963	0969	0976	0983	0990	0996	1003	1010	1017
6	1023	1030	1037	1043	1050	1057	1064	1070	1077	1084
7	1090	1097	1104	1111	1117	1124	1131	1137	1144	1151
8	1158	1164	1171	1178	1184	1191	1198	1204	1211	1218
9	1224	1231	1238	1245	1251	1258	1265	1271	1278	1285
650	81291	81298	81305	81311	81318	81325	81331	81338	81345	81351

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650	81291	81298	81305	81311	81318	81325	81331	81338	81345	81351
1	1358	1365	1371	1378	1385	1391	1398	1405	1411	1418
2	1425	1431	1438	1445	1451	1458	1465	1471	1478	1485
3	1491	1498	1505	1511	1518	1525	1531	1538	1544	1551
4	1558	1564	1571	1578	1584	1591	1598	1604	1611	1617
5	1624	1631	1637	1644	1651	1657	1664	1671	1677	1684
6	1690	1697	1704	1710	1717	1723	1730	1737	1743	1750
7	1757	1763	1770	1776	1783	1790	1796	1803	1809	1816
8	1823	1829	1836	1842	1849	1856	1862	1869	1875	1882
9	1889	1895	1902	1908	1915	1921	1928	1935	1941	1948
660	81954	81961	81968	81974	81981	81987	81994	82000	82007	82014
1	2020	2027	2033	2040	2046	2053	2060	2066	2073	2079
2	2086	2092	2099	2105	2112	2119	2125	2132	2138	2145
3	2151	2158	2164	2171	2178	2184	2191	2197	2204	2210
4	2217	2223	2230	2236	2243	2249	2256	2263	2269	2276
5	2282	2289	2295	2302	2308	2315	2321	2328	2334	2341
6	2347	2354	2360	2367	2373	2380	2387	2393	2400	2406
7	2413	2419	2426	2432	2439	2445	2452	2458	2465	2471
8	2478	2484	2491	2497	2504	2510	2517	2523	2530	2536
9	2543	2549	2556	2562	2569	2575	2582	2588	2595	2601
670	82607	82614	82620	82627	82633	82640	82646	82653	82659	82666
1	2672	2679	2685	2692	2698	2705	2711	2718	2724	2730
2	2737	2743	2750	2756	2763	2769	2776	2782	2789	2795
3	2802	2808	2814	2821	2827	2834	2840	2847	2853	2860
4	2866	2872	2879	2885	2892	2898	2905	2911	2918	2924
5	2930	2937	2943	2950	2956	2963	2969	2975	2982	2988
6	2995	3001	3008	3014	3020	3027	3033	3040	3046	3052
7	3059	3065	3072	3078	3085	3091	3097	3104	3110	3117
8	3123	3129	3136	3142	3149	3155	3161	3168	3174	3181
9	3187	3193	3200	3206	3213	3219	3225	3232	3238	3245
680	83251	83257	83264	83270	83276	83283	83289	83296	83302	83308
1	3315	3321	3327	3334	3340	3347	3353	3359	3366	3372
2	3378	3385	3391	3398	3404	3410	3417	3423	3429	3436
3	3442	3448	3455	3461	3467	3474	3480	3487	3493	3499
4	3506	3512	3518	3525	3531	3537	3544	3550	3556	3563
5	3569	3575	3582	3588	3594	3601	3607	3613	3620	3626
6	3632	3639	3645	3651	3658	3664	3670	3677	3683	3689
7	3696	3702	3708	3715	3721	3727	3734	3740	3746	3753
8	3759	3765	3771	3778	3784	3790	3797	3803	3809	3816
9	3822	3828	3835	3841	3847	3853	3860	3866	3872	3879
690	83885	83891	83897	83904	83910	83916	83923	83929	83935	83942
1	3948	3954	3960	3967	3973	3979	3985	3992	3998	4004
2	4011	4017	4023	4029	4036	4042	4048	4055	4061	4067
3	4073	4080	4086	4092	4098	4105	4111	4117	4123	4130
4	4136	4142	4148	4155	4161	4167	4173	4180	4186	4192
5	4198	4205	4211	4217	4223	4230	4236	4242	4248	4255
6	4261	4267	4273	4280	4286	4292	4298	4305	4311	4317
7	4323	4330	4336	4342	4348	4354	4361	4367	4373	4379
8	4386	4392	4398	4404	4410	4417	4423	4429	4435	4442
9	4448	4454	4460	4466	4473	4479	4485	4491	4497	4504
700	84510	84516	84522	84528	84535	84541	84547	84553	84559	84566

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700	84510	84516	84522	84528	84535	84541	84547	84553	84559	84566
1	4572	4578	4584	4590	4597	4603	4609	4615	4621	4628
2	4634	4640	4646	4652	4658	4665	4671	4677	4683	4689
3	4696	4702	4708	4714	4720	4726	4733	4739	4745	4751
4	4757	4763	4770	4776	4782	4788	4794	4800	4807	4813
5	4819	4825	4831	4837	4844	4850	4856	4862	4868	4874
6	4880	4887	4893	4899	4905	4911	4917	4924	4930	4936
7	4942	4948	4954	4960	4967	4973	4979	4985	4991	4997
8	5003	5009	5016	5022	5028	5034	5040	5046	5052	5058
9	5065	5071	5077	5083	5089	5095	5101	5107	5114	5120
710	85126	85132	85138	85144	85150	85156	85163	85169	85175	85181
1	5187	5193	5199	5205	5211	5217	5224	5230	5236	5242
2	5248	5254	5260	5266	5272	5278	5285	5291	5297	5303
3	5309	5315	5321	5327	5333	5339	5345	5352	5358	5364
4	5370	5376	5382	5388	5394	5400	5406	5412	5418	5425
5	5431	5437	5443	5449	5455	5461	5467	5473	5479	5485
6	5491	5497	5503	5509	5516	5522	5528	5534	5540	5546
7	5552	5558	5564	5570	5576	5582	5588	5594	5600	5606
8	5612	5618	5625	5631	5637	5643	5649	5655	5661	5667
9	5673	5679	5685	5691	5697	5703	5709	5715	5721	5727
720	85733	85739	85745	85751	85757	85763	85769	85775	85781	85788
1	5794	5800	5806	5812	5818	5824	5830	5836	5842	5848
2	5854	5860	5866	5872	5878	5884	5890	5896	5902	5908
3	5914	5920	5926	5932	5938	5944	5950	5956	5962	5968
4	5974	5980	5986	5992	5998	6004	6010	6016	6022	6028
5	6034	6040	6046	6052	6058	6064	6070	6076	6082	6088
6	6094	6100	6106	6112	6118	6124	6130	6136	6141	6147
7	6153	6159	6165	6171	6177	6183	6189	6195	6201	6207
8	6213	6219	6225	6231	6237	6243	6249	6255	6261	6267
9	6273	6279	6285	6291	6297	6303	6308	6314	6320	6326
730	86332	86338	86344	86350	86356	86362	86368	86374	86380	86386
1	6392	6398	6404	6410	6415	6421	6427	6433	6439	6445
2	6451	6457	6463	6469	6475	6481	6487	6493	6499	6504
3	6510	6516	6522	6528	6534	6540	6546	6552	6558	6564
4	6570	6576	6581	6587	6593	6599	6605	6611	6617	6623
5	6629	6635	6641	6646	6652	6658	6664	6670	6676	6682
6	6688	6694	6700	6705	6711	6717	6723	6729	6735	6741
7	6747	6753	6759	6764	6770	6776	6782	6788	6794	6800
8	6806	6812	6817	6823	6829	6835	6841	6847	6853	6859
9	6864	6870	6876	6882	6888	6894	6900	6906	6911	6917
740	86923	86929	86935	86941	86947	86953	86958	86964	86970	86976
1	6982	6988	6994	6999	7005	7011	7017	7023	7029	7035
2	7040	7046	7052	7058	7064	7070	7075	7081	7087	7093
3	7099	7105	7111	7116	7122	7128	7134	7140	7146	7151
4	7157	7163	7169	7175	7181	7186	7192	7198	7204	7210
5	7216	7221	7227	7233	7239	7245	7251	7256	7262	7268
6	7274	7280	7286	7291	7297	7303	7309	7315	7320	7326
7	7332	7338	7344	7349	7355	7361	7367	7373	7379	7384
8	7390	7396	7402	7408	7413	7419	7425	7431	7437	7442
9	7448	7454	7460	7466	7471	7477	7483	7489	7495	7500
750	87506	87512	87518	87523	87529	87535	87541	87547	87552	87558

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750	87506	87512	87518	87523	87529	87535	87541	87547	87552	87558
1	7564	7570	7576	7581	7587	7593	7599	7604	7610	7616
2	7622	7628	7633	7639	7645	7651	7656	7662	7668	7674
3	7679	7685	7691	7697	7703	7708	7714	7720	7726	7731
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9	9520	9524	9528	9533	9537	9542	9546	9550	9555	9559
990	99564	99568	99572	99577	99581	99585	99590	99594	99599	99603
1	9607	9612	9616	9621	9625	9629	9634	9638	9642	9647
2	9651	9656	9660	9664	9669	9673	9677	9682	9686	9691
3	9695	9699	9704	9708	9712	9717	9721	9726	9730	9734
4	9739	9743	9747	9752	9756	9760	9765	9769	9774	9778
5	9782	9787	9791	9795	9800	9804	9808	9813	9817	9822
6	9826	9830	9835	9839	9843	9848	9852	9856	9861	9865
7	9870	9874	9878	9883	9887	9891	9896	9900	9904	9909
8	9913	9917	9922	9926	9930	9935	9939	9944	9948	9952
9	9957	9961	9965	9970	9974	9978	9983	9987	9991	9996
1000	00000	00004	00009	00013	00017	00022	00026	00030	00035	00039

′	0°		1°		2°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	— ∞	10.00000	8.24186	9.99993	8.54282	9.99974	60
1	6.46373	00000	24903	99993	54642	99973	59
2	76476	00000	25609	99993	54999	99973	58
3	94085	00000	26304	99993	55354	99972	57
4	7.06579	00000	26988	99992	55705	99972	56
5	16270	00000	27661	99992	56054	99971	55
6	24188	00000	28324	99992	56400	99971	54
7	30882	00000	28977	99992	56743	99970	53
8	36682	00000	29621	99992	57084	99970	52
9	41797	00000	30255	99991	57421	99969	51
10	7.46373	10.00000	8.30879	9.99991	8.57757	9.99969	50
11	50512	00000	31495	99991	58089	99968	49
12	54291	00000	32103	99990	58419	99968	48
13	57767	00000	32702	99990	58747	99967	47
14	60985	00000	33292	99990	59072	99967	46
15	63982	00000	33875	99990	59395	99967	45
16	66784	00000	34450	99989	59715	99966	44
17	69417	9.99999	35018	99989	60033	99966	43
18	71900	99999	35578	99989	60349	99965	42
19	74248	99999	36131	99989	60662	99964	41
20	7.76475	9.99999	8.36678	9.99988	8.60973	9.99964	40
21	78594	99999	37217	99988	61282	99963	39
22	80615	99999	37750	99988	61589	99963	38
23	82545	99999	38276	99987	61894	99962	37
24	84393	99999	38796	99987	62196	99962	36
25	86166	99999	39310	99987	62497	99961	35
26	87870	99999	39818	99986	62795	99961	34
27	89509	99999	40320	99986	63091	99960	33
28	91088	99999	40816	99986	63385	99960	32
29	92612	99998	41307	99985	63678	99959	31
30	7.94084	9.99998	8.41792	9.99985	8.63968	9.99959	30
31	95508	99998	42272	99985	64256	99958	29
32	96887	99998	42746	99984	64543	99958	28
33	98223	99998	43216	99984	64827	99957	27
34	99520	99998	43680	99984	65110	99956	26
35	8.00779	99998	44139	99983	65391	99956	25
36	02002	99998	44594	99983	65670	99955	24
37	03192	99997	45044	99983	65947	99955	23
38	04350	99997	45489	99982	66223	99954	22
39	05478	99997	45930	99982	66497	99954	21
40	8.06578	9.99997	8.46366	9.99982	8.66769	9.99953	20
41	07650	99997	46799	99981	67039	99952	19
42	08696	99997	47226	99981	67308	99952	18
43	09718	99997	47650	99981	67575	99951	17
44	10717	99996	48069	99980	67841	99951	16
45	11693	99996	48485	99980	68104	99950	15
46	12647	99996	48896	99979	68367	99949	14
47	13581	99996	49304	99979	68627	99949	13
48	14495	99996	49708	99979	68886	99948	12
49	15391	99996	50108	99978	69144	99948	11
50	8.16268	9.99995	8.50504	9.99978	8.69400	9.99947	10
51	17128	99995	50897	99977	69654	99946	9
52	17971	99995	51287	99977	69907	99946	8
53	18798	99995	51673	99977	70159	99945	7
54	19610	99995	52055	99976	70409	99944	6
55	20407	99994	52434	99976	70658	99944	5
56	21189	99994	52810	99975	70905	99943	4
57	21958	99994	53183	99975	71151	99942	3
58	22713	99994	53552	99974	71395	99942	2
59	23456	99994	53919	99974	71638	99941	1
60	24186	99993	54282	99974	71880	99940	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	89°		88°		87°		

′	3°		4°		5°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	8.71880	9.99940	8.84358	9.99894	8.94030	9.99834	60
1	72120	99940	84539	99893	94174	99833	59
2	72359	99939	84718	99892	94317	99832	58
3	72597	99938	84897	99891	94461	99831	57
4	72834	99938	85075	99891	94603	99830	56
5	73069	99937	85252	99890	94746	99829	55
6	73303	99936	85429	99889	94887	99828	54
7	73535	99936	85605	99888	95029	99827	53
8	73767	99935	85780	99887	95170	99825	52
9	73997	99934	85955	99886	95310	99824	51
10	8.74226	9.99934	8.86128	9.99885	8.95450	9.99823	50
11	74454	99933	86301	99884	95589	99822	49
12	74680	99932	86474	99883	95728	99821	48
13	74906	99932	86645	99882	95867	99820	47
14	75130	99931	86816	99881	96005	99819	46
15	75353	99930	86987	99880	96143	99817	45
16	75575	99929	87156	99879	96280	99816	44
17	75795	99929	87325	99879	96417	99815	43
18	76015	99928	87494	99878	96553	99814	42
19	76234	99927	87661	99877	96689	99813	41
20	8.76451	9.99926	8.87829	9.99876	8.96825	9.99812	40
21	76667	99926	87995	99875	96960	99810	39
22	76883	99925	88161	99874	97095	99809	38
23	77097	99924	88326	99873	97229	99808	37
24	77310	99923	88490	99872	97363	99807	36
25	77522	99923	88654	99871	97496	99806	35
26	77733	99922	88817	99870	97629	99804	34
27	77943	99921	88980	99869	97762	99803	33
28	78152	99920	89142	99868	97894	99802	32
29	78360	99920	89304	99867	98026	99801	31
30	8.78568	9.99919	8.89464	9.99866	8.98157	9.99800	30
31	78774	99918	89625	99865	98288	99798	29
32	78979	99917	89784	99864	98419	99797	28
33	79183	99917	89943	99863	98549	99796	27
34	79386	99916	90102	99862	98679	99795	26
35	79588	99915	90260	99861	98808	99793	25
36	79789	99914	90417	99860	98937	99792	24
37	79990	99913	90574	99859	99066	99791	23
38	80189	99913	90730	99858	99194	99790	22
39	80388	99912	90885	99857	99322	99788	21
40	8.80585	9.99911	8.91040	9.99856	8.99450	9.99787	20
41	80782	99910	91195	99855	99577	99786	19
42	80978	99909	91349	99854	99704	99785	18
43	81173	99909	91502	99853	99830	99783	17
44	81367	99908	91655	99852	99956	99782	16
45	81560	99907	91807	99851	9.00082	99781	15
46	81752	99906	91959	99850	00207	99780	14
47	81944	99905	92110	99848	00332	99778	13
48	82134	99904	92261	99847	00456	99777	12
49	82324	99904	92411	99846	00581	99776	11
50	8.82513	9.99903	8.92561	9.99845	9.00704	9.99775	10
51	82701	99902	92710	99844	00828	99773	9
52	82888	99901	92859	99843	00951	99772	8
53	83075	99900	93007	99842	01074	99771	7
54	83261	99899	93154	99841	01196	99769	6
55	83446	99898	93301	99840	01318	99768	5
56	83630	99898	93448	99839	01440	99767	4
57	83813	99897	93594	99838	01561	99765	3
58	83996	99896	93740	99837	01682	99764	2
59	84177	99895	93885	99836	01803	99763	1
60	84358	99894	94030	99834	01923	99761	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	86°		85°		84°		

′	6° Sine	6° Cosine	7° Sine	7° Cosine	8° Sine	8° Cosine	′
0	9.01923	9.99761	9.08589	9.99675	9.14356	9.99575	60
1	02043	99760	08692	99674	14445	99574	59
2	02163	99759	08795	99672	14535	99572	58
3	02283	99757	08897	99670	14624	99570	57
4	02402	99756	08999	99669	14714	99568	56
5	02520	99755	09101	99667	14803	99566	55
6	02639	99753	09202	99666	14891	99565	54
7	02757	99752	09304	99664	14980	99563	53
8	02874	99751	09405	99663	15069	99561	52
9	02992	99749	09506	99661	15157	99559	51
10	9.03109	9.99748	9.09606	9.99659	9.15245	9.99557	50
11	03226	99747	09707	99658	15333	99556	49
12	03342	99745	09807	99656	15421	99554	48
13	03458	99744	09907	99655	15508	99552	47
14	03574	99742	10006	99653	15596	99550	46
15	03690	99741	10106	99651	15683	99548	45
16	03805	99740	10205	99650	15770	99546	44
17	03920	99738	10304	99648	15857	99545	43
18	04034	99737	10402	99647	15944	99543	42
19	04149	99736	10501	99645	16030	99541	41
20	9.04262	9.99734	9.10599	9.99643	9.16116	9.99539	40
21	04376	99733	10697	99642	16203	99537	39
22	04490	99731	10795	99640	16289	99535	38
23	04603	99730	10893	99638	16374	99533	37
24	04715	99728	10990	99637	16460	99532	36
25	04828	99727	11087	99635	16545	99530	35
26	04940	99726	11184	99633	16631	99528	34
27	05052	99724	11281	99632	16716	99526	33
28	05164	99723	11377	99630	16801	99524	32
29	05275	99721	11474	99629	16886	99522	31
30	9.05386	9.99720	9.11570	9.99627	9.16970	9.99520	30
31	05497	99718	11666	99625	17055	99518	29
32	05607	99717	11761	99624	17139	99517	28
33	05717	99716	11857	99622	17223	99515	27
34	05827	99714	11952	99620	17307	99513	26
35	05937	99713	12047	99618	17391	99511	25
36	06046	99711	12142	99617	17474	99509	24
37	06155	99710	12236	99615	17558	99507	23
38	06264	99708	12331	99613	17641	99505	22
39	06372	99707	12425	99612	17724	99503	21
40	9.06481	9.99705	9.12519	9.99610	9.17807	9.99501	20
41	06589	99704	12612	99608	17890	99499	19
42	06696	99702	12706	99607	17973	99497	18
43	06804	99701	12799	99605	18055	99495	17
44	06911	99699	12892	99603	18137	99494	16
45	07018	99698	12985	99601	18220	99492	15
46	07124	99696	13078	99600	18302	99490	14
47	07231	99695	13171	99598	18383	99488	13
48	07337	99693	13263	99596	18465	99486	12
49	07442	99692	13355	99595	18547	99484	11
50	9.07548	9.99690	9.13447	9.99593	9.18628	9.99482	10
51	07653	99689	13539	99591	18709	99480	9
52	07758	99687	13630	99589	18790	99478	8
53	07863	99686	13722	99588	18871	99476	7
54	07968	99684	13813	99586	18952	99474	6
55	08072	99683	13904	99584	19033	99472	5
56	08176	99681	13994	99582	19113	99470	4
57	08280	99680	14085	99581	19193	99468	3
58	08383	99678	14175	99579	19273	99466	2
59	08486	99677	14266	99577	19353	99464	1
60	08589	99675	14356	99575	19433	99462	0
′	Cosine 83°	Sine 83°	Cosine 82°	Sine 82°	Cosine 81°	Sine 81°	′

′	9° Sine	9° Cosine	10° Sine	10° Cosine	11° Sine	11° Cosine	′
0	9.19433	9.99462	9.23967	9.99335	9.28060	9.99195	60
1	19513	99460	24039	99333	28125	99192	59
2	19592	99458	24110	99331	28190	99190	58
3	19672	99456	24181	99328	28254	99187	57
4	19751	99454	24253	99326	28319	99185	56
5	19830	99452	24324	99324	28384	99182	55
6	19909	99450	24395	99322	28448	99180	54
7	19988	99448	24466	99319	28512	99177	53
8	20067	99446	24536	99317	28577	99175	52
9	20145	99444	24607	99315	28641	99172	51
10	9.20223	9.99442	9.24677	9.99313	9.28705	9.99170	50
11	20302	99440	24748	99310	28769	99167	49
12	20380	99438	24818	99308	28833	99165	48
13	20458	99436	24888	99306	28896	99162	47
14	20535	99434	24958	99304	28960	99160	46
15	20613	99432	25028	99301	29024	99157	45
16	20691	99429	25098	99299	29087	99155	44
17	20768	99427	25168	99297	29150	99152	43
18	20845	99425	25237	99294	29214	99150	42
19	20922	99423	25307	99292	29277	99147	41
20	9.20999	9.99421	9.25376	9.99290	9.29340	9.99145	40
21	21076	99419	25445	99288	29403	99142	39
22	21153	99417	25514	99285	29466	99140	38
23	21229	99415	25583	99283	29529	99137	37
24	21306	99413	25652	99281	29591	99135	36
25	21382	99411	25721	99278	29654	99132	35
26	21458	99409	25790	99276	29716	99130	34
27	21534	99407	25858	99274	29779	99127	33
28	21610	99404	25927	99271	29841	99124	32
29	21685	99402	25995	99269	29903	99122	31
30	9.21761	9.99400	9.26063	9.99267	9.29966	9.99119	30
31	21836	99398	26131	99264	30028	99117	29
32	21912	99396	26199	99262	30090	99114	28
33	21987	99394	26267	99260	30151	99112	27
34	22062	99392	26335	99257	30213	99109	26
35	22137	99390	26403	99255	30275	99106	25
36	22211	99388	26470	99252	30336	99104	24
37	22286	99385	26538	99250	30398	99101	23
38	22361	99383	26605	99248	30459	99099	22
39	22435	99381	26672	99245	30521	99096	21
40	9.22509	9.99379	9.26739	9.99243	9.30582	9.99093	20
41	22583	99377	26806	99241	30643	99091	19
42	22657	99375	26873	99238	30704	99088	18
43	22731	99372	26940	99236	30765	99086	17
44	22805	99370	27007	99233	30826	99083	16
45	22878	99368	27073	99231	30887	99080	15
46	22952	99366	27140	99229	30947	99078	14
47	23025	99364	27206	99226	31008	99075	13
48	23098	99362	27273	99224	31068	99072	12
49	23171	99359	27339	99221	31129	99070	11
50	9.23244	9.99357	9.27405	9.99219	9.31189	9.99067	10
51	23317	99355	27471	99217	31250	99064	9
52	23390	99353	27537	99214	31310	99062	8
53	23462	99351	27602	99212	31370	99059	7
54	23535	99348	27668	99209	31430	99056	6
55	23607	99346	27734	99207	31490	99054	5
56	23679	99344	27799	99204	31549	99051	4
57	23752	99342	27864	99202	31609	99048	3
58	23823	99340	27930	99200	31669	99046	2
59	23895	99337	27995	99197	31728	99043	1
60	23967	99335	28060	99195	31788	99040	0
′	80° Cosine	80° Sine	79° Cosine	79° Sine	78° Cosine	78° Sine	′

′	12°		13°		14°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	9.31788	9.99040	9.35209	9.98872	9.38368	9.98690	60
1	31847	99038	35263	98869	38418	98687	59
2	31907	99035	35318	98867	38469	98684	58
3	31966	99032	35373	98864	38519	98681	57
4	32025	99030	35427	98861	38570	98678	56
5	32084	99027	35481	98858	38620	98675	55
6	32143	99024	35536	98855	38670	98671	54
7	32202	99022	35590	98852	38721	98668	53
8	32261	99019	35644	98849	38771	98665	52
9	32319	99016	35698	98846	38821	98662	51
10	9.32378	9.99013	9.35752	9.98843	9.38871	9.98659	50
11	32437	99011	35806	98840	38921	98656	49
12	32495	99008	35860	98837	38971	98652	48
13	32553	99005	35914	98834	39021	98649	47
14	32612	99002	35968	98831	39071	98646	46
15	32670	99000	36022	98828	39121	98643	45
16	32728	98997	36075	98825	39170	98640	44
17	32786	98994	36129	98822	39220	98636	43
18	32844	98991	36182	98819	39270	98633	42
19	32902	98989	36236	98816	39319	98630	41
20	9.32960	9.98986	9.36289	9.98813	9.39369	9.98627	40
21	33018	98983	36342	98810	39418	98623	39
22	33075	98980	36395	98807	39467	98620	38
23	33133	98978	36449	98804	39517	98617	37
24	33190	98975	36502	98801	39566	98614	36
25	33248	98972	36555	98798	39615	98610	35
26	33305	98969	36608	98795	39664	98607	34
27	33362	98967	36660	98792	39713	98604	33
28	33420	98964	36713	98789	39762	98601	32
29	33477	98961	36766	98786	39811	98597	31
30	9.33534	9.98958	9.36819	9.98783	9.39860	9.98594	30
31	33591	98955	36871	98780	39909	98591	29
32	33647	98953	36924	98777	39958	98588	28
33	33704	98950	36976	98774	40006	98584	27
34	33761	98947	37028	98771	40055	98581	26
35	33818	98944	37081	98768	40103	98578	25
36	33874	98941	37133	98765	40152	98574	24
37	33931	98938	37185	98762	40200	98571	23
38	33987	98936	37237	98759	40249	98568	22
39	34043	98933	37289	98756	40297	98565	21
40	9.34100	9.98930	9.37341	9.98753	9.40346	9.98561	20
41	34156	98927	37393	98750	40394	98558	19
42	34212	98924	37445	98746	40442	98555	18
43	34268	98921	37497	98743	40490	98551	17
44	34324	98919	37549	98740	40538	98548	16
45	34380	98916	37600	98737	40586	98545	15
46	34436	98913	37652	98734	40634	98541	14
47	34491	98910	37703	98731	40682	98538	13
48	34547	98907	37755	98728	40730	98535	12
49	34602	98904	37806	98725	40778	98531	11
50	9.34658	9.98901	9.37858	9.98722	9.40825	9.98528	10
51	34713	98898	37909	98719	40873	98525	9
52	34769	98896	37960	98715	40921	98521	8
53	34824	98893	38011	98712	40968	98518	7
54	34879	98890	38062	98709	41016	98515	6
55	34934	98887	38113	98706	41063	98511	5
56	34989	98884	38164	98703	41111	98508	4
57	35044	98881	38215	98700	41158	98505	3
58	35099	98878	38266	98697	41205	98501	2
59	35154	98875	38317	98694	41252	98498	1
60	35209	98872	38368	98690	41300	98494	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	77°		76°		75°		

′	15°		16°		17°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	9.41300	9.98494	9.44034	9.98284	9.46594	9.98060	**60**
1	41347	98491	44078	98281	46635	98056	**59**
2	41394	98488	44122	98277	46676	98052	**58**
3	41441	98484	44166	98273	46717	98048	**57**
4	41488	98481	44210	98270	46758	98044	**56**
5	41535	98477	44253	98266	46800	98040	**55**
6	41582	98474	44297	98262	46841	98036	**54**
7	41628	98471	44341	98259	46882	98032	**53**
8	41675	98467	44385	98255	46923	98029	**52**
9	41722	98464	44428	98251	46964	98025	**51**
10	9.41768	9.98460	9.44472	9.98248	9.47005	9.98021	**50**
11	41815	98457	44516	98244	47045	98017	**49**
12	41861	98453	44559	98240	47086	98013	**48**
13	41908	98450	44602	98237	47127	98009	**47**
14	41954	98447	44646	98233	47168	98005	**46**
15	42001	98443	44689	98229	47209	98001	**45**
16	42047	98440	44733	98226	47249	97997	**44**
17	42093	98436	44776	98222	47290	97993	**43**
18	42140	98433	44819	98218	47330	97989	**42**
19	42186	98429	44862	98215	47371	97986	**41**
20	9.42232	9.98426	9.44905	9.98211	9.47411	9.97982	**40**
21	42278	98422	44948	98207	47452	97978	**39**
22	42324	98419	44992	98204	47492	97974	**38**
23	42370	98415	45035	98200	47533	97970	**37**
24	42416	98412	45077	98196	47573	97966	**36**
25	42461	98409	45120	98192	47613	97962	**35**
26	42507	98405	45163	98189	47654	97958	**34**
27	42553	98402	45206	98185	47694	97954	**33**
28	42599	98398	45249	98181	47734	97950	**32**
29	42644	98395	45292	98177	47774	97946	**31**
30	9.42690	9.98391	9.45334	9.98174	9.47814	9.97942	**30**
31	42735	98388	45377	98170	47854	97938	**29**
32	42781	98384	45419	98166	47894	97934	**28**
33	42826	98381	45462	98162	47934	97930	**27**
34	42872	98377	45504	98159	47974	97926	**26**
35	42917	98373	45547	98155	48014	97922	**25**
36	42962	98370	45589	98151	48054	97918	**24**
37	43008	98366	45632	98147	48094	97914	**23**
38	43053	98363	45674	98144	48133	97910	**22**
39	43098	98359	45716	98140	48173	97906	**21**
40	9.43143	9.98356	9.45758	9.98136	9.48213	9.97902	**20**
41	43188	98352	45801	98132	48252	97898	**19**
42	43233	98349	45843	98129	48292	97894	**18**
43	43278	98345	45885	98125	48332	97890	**17**
44	43323	98342	45927	98121	48371	97886	**16**
45	43367	98338	45969	98117	48411	97882	**15**
46	43412	98334	46011	98113	48450	97878	**14**
47	43457	98331	46053	98110	48490	97874	**13**
48	43502	98327	46095	98106	48529	97870	**12**
49	43546	98324	46136	98102	48568	97866	**11**
50	9.43591	9.98320	9.46178	9.98098	9.48607	9.97861	**10**
51	43635	98317	46220	98094	48647	97857	**9**
52	43680	98313	46262	98090	48686	97853	**8**
53	43724	98309	46303	98087	48725	97849	**7**
54	43769	98306	46345	98083	48764	97845	**6**
55	43813	98302	46386	98079	48803	97841	**5**
56	43857	98299	46428	98075	48842	97837	**4**
57	43901	98295	46469	98071	48881	97833	**3**
58	43946	98291	46511	98067	48920	97829	**2**
59	43990	98288	46552	98063	48959	97825	**1**
60	44034	98284	46594	98060	48998	97821	**0**
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	74°		73°		72°		

′	18°		19°		20°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	9.48998	9.97821	9.51264	9.97567	9.53405	9.97299	60
1	49037	97817	51301	97563	53440	97294	59
2	49076	97812	51338	97558	53475	97289	58
3	49115	97808	51374	97554	53509	97285	57
4	49153	97804	51411	97550	53544	97280	56
5	49192	97800	51447	97545	53578	97276	55
6	49231	97796	51484	97541	53613	97271	54
7	49269	97792	51520	97536	53647	97266	53
8	49308	97788	51557	97532	53682	97262	52
9	49347	97784	51593	97528	53716	97257	51
10	9.49385	9.97779	9.51629	9.97523	9.53751	9.97252	50
11	49424	97775	51666	97519	53785	97248	49
12	49462	97771	51702	97515	53819	97243	48
13	49500	97767	51738	97510	53854	97238	47
14	49539	97763	51774	97506	53888	97234	46
15	49577	97759	51811	97501	53922	97229	45
16	49615	97754	51847	97497	53957	97224	44
17	49654	97750	51883	97492	53991	97220	43
18	49692	97746	51919	97488	54025	97215	42
19	49730	97742	51955	97484	54059	97210	41
20	9.49768	9.97738	9.51991	9.97479	9.54093	9.97206	40
21	49806	97734	52027	97475	54127	97201	39
22	49844	97729	52063	97470	54161	97196	38
23	49882	97725	52099	97466	54195	97192	37
24	49920	97721	52135	97461	54229	97187	36
25	49958	97717	52171	97457	54263	97182	35
26	49996	97713	52207	97453	54297	97178	34
27	50034	97708	52242	97448	54331	97173	33
28	50072	97704	52278	97444	54365	97168	32
29	50110	97700	52314	97439	54399	97163	31
30	9.50148	9.97696	9.52350	9.97435	9.54433	9.97159	30
31	50185	97691	52385	97430	54466	97154	29
32	50223	97687	52421	97426	54500	97149	28
33	50261	97683	52456	97421	54534	97145	27
34	50298	97679	52492	97417	54567	97140	26
35	50336	97674	52527	97412	54601	97135	25
36	50374	97670	52563	97408	54635	97130	24
37	50411	97666	52598	97403	54668	97126	23
38	50449	97662	52634	97399	54702	97121	22
39	50486	97657	52669	97394	54735	97116	21
40	9.50523	9.97653	9.52705	9.97390	9.54769	9.97111	20
41	50561	97649	52740	97385	54802	97107	19
42	50598	97645	52775	97381	54836	97102	18
43	50635	97640	52811	97376	54869	97097	17
44	50673	97636	52846	97372	54903	97092	16
45	50710	97632	52881	97367	54936	97087	15
46	50747	97628	52916	97363	54969	97083	14
47	50784	97623	52951	97358	55003	97078	13
48	50821	97619	52986	97353	55036	97073	12
49	50858	97615	53021	97349	55069	97068	11
50	9.50896	9.97610	9.53056	9.97344	9.55102	9.97063	10
51	50933	97606	53092	97340	55136	97059	9
52	50970	97602	53126	97335	55169	97054	8
53	51007	97597	53161	97331	55202	97049	7
54	51043	97593	53196	97326	55235	97044	6
55	51080	97589	53231	97322	55268	97039	5
56	51117	97584	53266	97317	55301	97035	4
57	51154	97580	53301	97312	55334	97030	3
58	51191	97576	53336	97308	55367	97025	2
59	51227	97571	53370	97303	55400	97020	1
60	51264	97567	53405	97299	55433	97015	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	71°		70°		69°		

′	21° Sine	21° Cosine	22° Sine	22° Cosine	23° Sine	23° Cosine	′
0	9.55433	9.97015	9.57358	9.96717	9.59188	9.96403	60
1	55466	97010	57389	96711	59218	96397	59
2	55499	97005	57420	96706	59247	96392	58
3	55532	97001	57451	96701	59277	96387	57
4	55564	96996	57482	96696	59307	96381	56
5	55597	96991	57514	96691	59336	96376	55
6	55630	96986	57545	96686	59366	96370	54
7	55663	96981	57576	96681	59396	96365	53
8	55695	96976	57607	96676	59425	96360	52
9	55728	96971	57638	96670	59455	96354	51
10	9.55761	9.96966	9.57669	9.96665	9.59484	9.96349	50
11	55793	96962	57700	96660	59514	96343	49
12	55826	96957	57731	96655	59543	96338	48
13	55858	96952	57762	96650	59573	96333	47
14	55891	96947	57793	96645	59602	96327	46
15	55923	96942	57824	96640	59632	96322	45
16	55956	96937	57855	96634	59661	96316	44
17	55988	96932	57885	96629	59690	96311	43
18	56021	96927	57916	96624	59720	96305	42
19	56053	96922	57947	96619	59749	96300	41
20	9.56085	9.96917	9.57978	9.96614	9.59778	9.96294	40
21	56118	96912	58008	96608	59808	96289	39
22	56150	96907	58039	96603	59837	96284	38
23	56182	96903	58070	96598	59866	96278	37
24	56215	96898	58101	96593	59895	96273	36
25	56247	96893	58131	96588	59924	96267	35
26	56279	96888	58162	96582	59954	96262	34
27	56311	96883	58192	96577	59983	96256	33
28	56343	96878	58223	96572	60012	96251	32
29	56375	96873	58253	96567	60041	96245	31
30	9.56408	9.96868	9.58284	9.96562	9.60070	9.96240	30
31	56440	96863	58314	96556	60099	96234	29
32	56472	96858	58345	96551	60128	96229	28
33	56504	96853	58375	96546	60157	96223	27
34	56536	96848	58406	96541	60186	96218	26
35	56568	96843	58436	96535	60215	96212	25
36	56599	96838	58467	96530	60244	96207	24
37	56631	96833	58497	96525	60273	96201	23
38	56663	96828	58527	96520	60302	96196	22
39	56695	96823	58557	96514	60331	96190	21
40	9.56727	9.96818	9.58588	9.96509	9.60359	9.96185	20
41	56759	96813	58618	96504	60388	96179	19
42	56790	96808	58648	96498	60417	96174	18
43	56822	96803	58678	96493	60446	96168	17
44	56854	96798	58709	96488	60474	96162	16
45	56886	96793	58739	96483	60503	96157	15
46	56917	96788	58769	96477	60532	96151	14
47	56949	96783	58799	96472	60561	96146	13
48	56980	96778	58829	96467	60589	96140	12
49	57012	96772	58859	96461	60618	96135	11
50	9.57044	9.96767	9.58889	9.96456	9.60646	9.96129	10
51	57075	96762	58919	96451	60675	96123	9
52	57107	96757	58949	96445	60704	96118	8
53	57138	96752	58979	96440	60732	96112	7
54	57169	96747	59009	96435	60761	96107	6
55	57201	96742	59039	96429	60789	96101	5
56	57232	96737	59069	96424	60818	96095	4
57	57264	96732	59098	96419	60846	96090	3
58	57295	96727	59128	96413	60875	96084	2
59	57326	96722	59158	96408	60903	96079	1
60	57358	96717	59188	96403	60931	96073	0
′	68° Cosine	68° Sine	67° Cosine	67° Sine	66° Cosine	66° Sine	′

′	24° Sine	24° Cosine	25° Sine	25° Cosine	26° Sine	26° Cosine	′
0	9.60931	9.96073	9.62595	9.95728	9.64184	9.95366	60
1	60960	96067	62622	95722	64210	95360	59
2	60988	96062	62649	95716	64236	95354	58
3	61016	96056	62676	95710	64262	95348	57
4	61045	96050	62703	95704	64288	95341	56
5	61073	96045	62730	95698	64313	95335	55
6	61101	96039	62757	95692	64339	95329	54
7	61129	96034	62784	95686	64365	95323	53
8	61158	96028	62811	95680	64391	95317	52
9	61186	96022	62838	95674	64417	95310	51
10	9.61214	9.96017	9.62865	9.95668	9.64442	9.95304	50
11	61242	96011	62892	95663	64468	95298	49
12	61270	96005	62918	95657	64494	95292	48
13	61298	96000	62945	95651	64519	95286	47
14	61326	95994	62972	95645	64545	95279	46
15	61354	95988	62999	95639	64571	95273	45
16	61382	95982	63026	95633	64596	95267	44
17	61411	95977	63052	95627	64622	95261	43
18	61438	95971	63079	95621	64647	95254	42
19	61466	95965	63106	95615	64673	95248	41
20	9.61494	9.95960	9.63133	9.95609	9.64698	9.95242	40
21	61522	95954	63159	95603	64724	95236	39
22	61550	95948	63186	95597	64749	95229	38
23	61578	95942	63213	95591	64775	95223	37
24	61606	95937	63239	95585	64800	95217	36
25	61634	95931	63266	95579	64826	95211	35
26	61662	95925	63292	95573	64851	95204	34
27	61689	95920	63319	95567	64877	95198	33
28	61717	95914	63345	95561	64902	95192	32
29	61745	95908	63372	95555	64927	95185	31
30	9.61773	9.95902	9.63398	9.95549	9.64953	9.95179	30
31	61800	95897	63425	95543	64978	95173	29
32	61828	95891	63451	95537	65003	95167	28
33	61856	95885	63478	95531	65029	95160	27
34	61883	95879	63504	95525	65054	95154	26
35	61911	95873	63531	95519	65079	95148	25
36	61939	95868	63557	95513	65104	95141	24
37	61966	95862	63583	95507	65130	95135	23
38	61994	95856	63610	95500	65155	95129	22
39	62021	95850	63636	95494	65180	95122	21
40	9.62049	9.95844	9.63662	9.95488	9.65205	9.95116	20
41	62076	95839	63689	95482	65230	95110	19
42	62104	95833	63715	95476	65255	95103	18
43	62131	95827	63741	95470	65281	95097	17
44	62159	95821	63767	95464	65306	95090	16
45	62186	95815	63794	95458	65331	95084	15
46	62214	95810	63820	95452	65356	95078	14
47	62241	95804	63846	95446	65381	95071	13
48	62268	95798	63872	95440	65406	95065	12
49	62296	95792	63898	95434	65431	95059	11
50	9.62323	9.95786	9.63924	9.95427	9.65456	9.95052	10
51	62350	95780	63950	95421	65481	95046	9
52	62377	95775	63976	95415	65506	95039	8
53	62405	95769	64002	95409	65531	95033	7
54	62432	95763	64028	95403	65555	95027	6
55	62459	95757	64054	95397	65580	95020	5
56	62486	95751	64080	95391	65605	95014	4
57	62513	95745	64106	95384	65630	95007	3
58	62541	95739	64132	95378	65655	95001	2
59	62568	95733	64158	95372	65680	94995	1
60	62595	95728	64184	95366	65705	94988	0
′	65° Cosine	65° Sine	64° Cosine	64° Sine	63° Cosine	63° Sine	′

′	27° Sine	27° Cosine	28° Sine	28° Cosine	29° Sine	29° Cosine	′
0	9.65705	9.94988	9.67161	9.94593	9.68557	9.94182	60
1	65729	94982	67185	94587	68580	94175	59
2	65754	94975	67208	94580	68603	94168	58
3	65779	94969	67232	94573	68625	94161	57
4	65804	94962	67256	94567	68648	94154	56
5	65828	94956	67280	94560	68671	94147	55
6	65853	94949	67303	94553	68694	94140	54
7	65878	94943	67327	94546	68716	94133	53
8	65902	94936	67350	94540	68739	94126	52
9	65927	94930	67374	94533	68762	94119	51
10	9.65952	9.94923	9.67398	9.94526	9.68784	9.94112	50
11	65976	94917	67421	94519	68807	94105	49
12	66001	94911	67445	94513	68829	94098	48
13	66025	94904	67468	94506	68852	94090	47
14	66050	94898	67492	94499	68875	94083	46
15	66075	94891	67515	94492	68897	94076	45
16	66099	94885	67539	94485	68920	94069	44
17	66124	94878	67562	94479	68942	94062	43
18	66148	94871	67586	94472	68965	94055	42
19	66173	94865	67609	94465	68987	94048	41
20	9.66197	9.94858	9.67633	9.94458	9.69010	9.94041	40
21	66221	94852	67656	94451	69032	94034	39
22	66246	94845	67680	94445	69055	94027	38
23	66270	94839	67703	94438	69077	94020	37
24	66295	94832	67726	94431	69100	94012	36
25	66319	94826	67750	94424	69122	94005	35
26	66343	94819	67773	94417	69144	93998	34
27	66368	94813	67796	94410	69167	93991	33
28	66392	94806	67820	94404	69189	93984	32
29	66416	94799	67843	94397	69212	93977	31
30	9.66441	9.94793	9.67866	9.94390	9.69234	9.93970	30
31	66465	94786	67890	94383	69256	93963	29
32	66489	94780	67913	94376	69279	93955	28
33	66513	94773	67936	94369	69301	93948	27
34	66537	94767	67959	94362	69323	93941	26
35	66562	94760	67982	94355	69345	93934	25
36	66586	94753	68006	94349	69368	93927	24
37	66610	94747	68029	94342	69390	93920	23
38	66634	94740	68052	94335	69412	93912	22
39	66658	94734	68075	94328	69434	93905	21
40	9.66682	9.94727	9.68098	9.94321	9.69456	9.93898	20
41	66706	94720	68121	94314	69479	93891	19
42	66731	94714	68144	94307	69501	93884	18
43	66755	94707	68167	94300	69523	93876	17
44	66779	94700	68190	94293	69545	93869	16
45	66803	94694	68213	94286	69567	93862	15
46	66827	94687	68237	94279	69589	93855	14
47	66851	94680	68260	94273	69611	93847	13
48	66875	94674	68283	94266	69633	93840	12
49	66899	94667	68305	94259	69655	93833	11
50	9.66922	9.94660	9.68328	9.94252	9.69677	9.93826	10
51	66946	94654	68351	94245	69699	93819	9
52	66970	94647	68374	94238	69721	93811	8
53	66994	94640	68397	94231	69743	93804	7
54	67018	94634	68420	94224	69765	93797	6
55	67042	94627	68443	94217	69787	93789	5
56	67066	94620	68466	94210	69809	93782	4
57	67090	94614	68489	94203	69831	93775	3
58	67113	94607	68512	94196	69853	93768	2
59	67137	94600	68534	94189	69875	93760	1
60	67161	94593	68557	94182	69897	93753	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	62°		61°		60°		

′	30°		31°		32°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	9.69897	9.93753	9.71184	9.93307	9.72421	9.92842	60
1	69919	93746	71205	93299	72441	92834	59
2	69941	93738	71226	93291	72461	92826	58
3	69963	93731	71247	93284	72482	92818	57
4	69984	93724	71268	93276	72502	92810	56
5	70006	93717	71289	93269	72522	92803	55
6	70028	93709	71310	93261	72542	92795	54
7	70050	93702	71331	93253	72562	92787	53
8	70072	93695	71352	93246	72582	92779	52
9	70093	93687	71373	93238	72602	92771	51
10	9.70115	9.93680	9.71393	9.93230	9.72622	9.92763	50
11	70137	93673	71414	93223	72043	92755	49
12	70159	93665	71435	93215	72663	92747	48
13	70180	93658	71456	93207	72683	92739	47
14	70202	93650	71477	93200	72703	92731	46
15	70224	93643	71498	93192	72723	92723	45
16	70245	93636	71519	93184	72743	92715	44
17	70267	93628	71539	93177	72763	92707	43
18	70288	93621	71560	93169	72783	92699	42
19	70310	93614	71581	93161	72803	92691	41
20	9.70332	9.93606	9.71602	9.93154	9.72823	9.92683	40
21	70353	93599	71622	93146	72843	92675	39
22	70375	93591	71643	93138	72863	92667	38
23	70396	93584	71664	93131	72883	92659	37
24	70418	93577	71685	93123	72902	92651	36
25	70439	93569	71705	93115	72922	92643	35
26	70461	93562	71726	93108	72942	92635	34
27	70482	93554	71747	93100	72962	92627	33
28	70504	93547	71767	93092	72982	92619	32
29	70525	93539	71788	93084	73002	92611	31
30	9.70547	9.93532	9.71809	9.93077	9.73022	9.92603	30
31	70568	93525	71829	93069	73041	92595	29
32	70590	93517	71850	93061	73061	92587	28
33	70611	93510	71870	93053	73081	92579	27
34	70633	93502	71891	93046	73101	92571	26
35	70654	93495	71911	93038	73121	92563	25
36	70675	93487	71932	93030	73140	92555	24
37	70697	93480	71952	93022	73160	92546	23
38	70718	93472	71973	93014	73180	92538	22
39	70739	93465	71994	93007	73200	92530	21
40	9.70761	9.93457	9.72014	9.92999	9.73219	9.92522	20
41	70782	93450	72034	92991	73239	92514	19
42	70803	93442	72055	92983	73259	92506	18
43	70824	93435	72075	92976	73278	92498	17
44	70846	93427	72096	92968	73298	92490	16
45	70867	93420	72116	92960	73318	92482	15
46	70888	93412	72137	92952	73337	92473	14
47	70909	93405	72157	92944	73357	92465	13
48	70931	93397	72177	92936	73377	92457	12
49	70952	93390	72198	92929	73396	92449	11
50	9.70973	9.93382	9.72218	9.92921	9.73416	9.92441	10
51	70994	93375	72238	92913	73435	92433	9
52	71015	93367	72259	92905	73455	92425	8
53	71036	93360	72279	92897	73474	92416	7
54	71058	93352	72299	92889	73494	92408	6
55	71079	93344	72320	92881	73513	92400	5
56	71100	93337	72340	92874	73533	92392	4
57	71121	93329	72360	92866	73552	92384	3
58	71142	93322	72381	92858	73572	92376	2
59	71163	93314	72401	92850	73591	92367	1
60	71184	93307	72421	92842	73611	92359	0
	Cosine	Sine	Cosine	Sine	Cosine	Sine	
′	59°		58°		57°		′

TABLE XI LOGARITHMIC SINES AND COSINES *(Continued)*

′	33° Sine	33° Cosine	34° Sine	34° Cosine	35° Sine	35° Cosine	′
0	9.73611	9.92359	9.74756	9.91857	9.75859	9.91336	60
1	73630	92351	74775	91849	75877	91328	59
2	73650	92343	74794	91840	75895	91319	58
3	73669	92335	74812	91832	75913	91310	57
4	73689	92326	74831	91823	75931	91301	56
5	73708	92318	74850	91815	75949	91292	55
6	73727	92310	74868	91806	75967	91283	54
7	73747	92302	74887	91798	75985	91274	53
8	73766	92293	74906	91789	76003	91266	52
9	73785	92285	74924	91781	76021	91257	51
10	9.73805	9.92277	9.74943	9.91772	9.76039	9.91248	50
11	73824	92269	74961	91763	76057	91239	49
12	73843	92260	74980	91755	76075	91230	48
13	73863	92252	74999	91746	76093	91221	47
14	73882	92244	75017	91738	76111	91212	46
15	73901	92235	75036	91729	76129	91203	45
16	73921	92227	75054	91720	76146	91194	44
17	73940	92219	75073	91712	76164	91185	43
18	73959	92211	75091	91703	76182	91176	42
19	73978	92202	75110	91695	76200	91167	41
20	9.73997	9.92194	9.75128	9.91686	9.76218	9.91158	40
21	74017	92186	75147	91677	76236	91149	39
22	74036	92177	75165	91669	76253	91141	38
23	74055	92169	75184	91660	76271	91132	37
24	74074	92161	75202	91651	76289	91123	36
25	74093	92152	75221	91643	76307	91114	35
26	74113	92144	75239	91634	76324	91105	34
27	74132	92136	75258	91625	76342	91096	33
28	74151	92127	75276	91617	76360	91087	32
29	74170	92119	75294	91608	76378	91078	31
30	9.74189	9.92111	9.75313	9.91599	9.76395	9.91069	30
31	74208	92102	75331	91591	76413	91060	29
32	74227	92094	75350	91582	76431	91051	28
33	74246	92086	75368	91573	76448	91042	27
34	74265	92077	75386	91565	76466	91033	26
35	74284	92069	75405	91556	76484	91023	25
36	74303	92060	75423	91547	76501	91014	24
37	74322	92052	75441	91538	76519	91005	23
38	74341	92044	75459	91530	76537	90996	22
39	74360	92035	75478	91521	76554	90987	21
40	9.74379	9.92027	9.75496	9.91512	9.76572	9.90978	20
41	74398	92018	75514	91504	76590	90969	19
42	74417	92010	75533	91495	76607	90960	18
43	74436	92002	75551	91486	76625	90951	17
44	74455	91993	75569	91477	76642	90942	16
45	74474	91985	75587	91469	76660	90933	15
46	74493	91976	75605	91460	76677	90924	14
47	74512	91968	75624	91451	76695	90915	13
48	74531	91959	75642	91442	76712	90906	12
49	74549	91951	75660	91433	76730	90896	11
50	9.74568	9.91942	9.75678	9.91425	9.76747	9.90887	10
51	74587	91934	75696	91416	76765	90878	9
52	74606	91925	75714	91407	76782	90869	8
53	74625	91917	75733	91398	76800	90860	7
54	74644	91908	75751	91389	76817	90851	6
55	74662	91900	75769	91381	76835	90842	5
56	74681	91891	75787	91372	76852	90832	4
57	74700	91883	75805	91363	76870	90823	3
58	74719	91874	75823	91354	76887	90814	2
59	74737	91866	75841	91345	76904	90805	1
60	74756	91857	75859	91336	76922	90796	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	56°		55°		54°		

′	36°		37°		38°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	9.76922	9.90796	9.77946	9.90235	9.78934	9.89653	60
1	76939	90787	77963	90225	78950	89643	59
2	76957	90777	77980	90216	78967	89633	58
3	76974	90768	77997	90206	78983	89624	57
4	76991	90759	78013	90197	78999	89614	56
5	77009	90750	78030	90187	79015	89604	55
6	77026	90741	78047	90178	79031	89594	54
7	77043	90731	78063	90168	79047	89584	53
8	77061	90722	78080	90159	79063	89574	52
9	77078	90713	78097	90149	79079	89564	51
10	9.77095	9.90704	9.78113	9.90139	9.79095	9.89554	50
11	77112	90694	78130	90130	79111	89544	49
12	77130	90685	78147	90120	79128	89534	48
13	77147	90676	78163	90111	79144	89524	47
14	77164	90667	78180	90101	79160	89514	46
15	77181	90657	78197	90091	79176	89504	45
16	77199	90648	78218	90082	79192	89495	44
17	77216	90639	78230	90072	79208	89485	43
18	77233	90630	78246	90063	79224	89475	42
19	77250	90620	78263	90053	79240	89465	41
20	9.77268	9.90611	9.78280	9.90043	9.79256	9.89455	40
21	77285	90602	78296	90034	79272	89445	39
22	77302	90592	78313	90024	79288	89435	38
23	77319	90583	78329	90014	79304	89425	37
24	77336	90574	78346	90005	79319	89415	36
25	77353	90565	78362	89995	79335	89405	35
26	77370	90555	78379	89985	79351	89395	34
27	77387	90546	78395	89976	79367	89385	33
28	77405	90537	78412	89966	79383	89375	32
29	77422	90527	78428	89956	79399	89364	31
30	9.77439	9.90518	9.78445	9.89947	9.79415	9.89354	30
31	77456	90509	78461	89937	79431	89344	29
32	77473	90499	78478	89927	79447	89334	28
33	77490	90490	78494	89918	79463	89324	27
34	77507	90480	78510	89908	79478	89314	26
35	77524	90471	78527	89898	79494	89304	25
36	77541	90462	78543	89888	79510	89294	24
37	77558	90452	78560	89879	79526	89284	23
38	77575	90443	78576	89869	79542	89274	22
39	77592	90434	78592	89859	79558	89264	21
40	9.77609	9.90424	9.78609	9.89849	9.79573	9.89254	20
41	77626	90415	78625	89840	79589	89244	19
42	77643	90405	78642	89830	79605	89233	18
43	77660	90396	78658	89820	78621	89223	17
44	77677	90386	78674	89810	79636	89213	16
45	77694	90377	78691	89801	79652	89203	15
46	77711	90368	78707	89791	79668	89193	14
47	77728	90358	78723	89781	79684	89183	13
48	77744	90349	78739	89771	79699	89173	12
49	77761	90339	78756	89761	79715	89162	11
50	9.77778	9.90330	9.78772	9.89752	9.79731	9.89152	10
51	77795	90320	78788	89742	79746	89142	9
52	77812	90311	78805	89732	79762	89132	8
53	77829	90301	78821	89722	79778	89122	7
54	77846	90292	78837	89712	79793	89112	6
55	77862	90282	78853	89702	79809	89101	5
56	77879	90273	78869	89693	79825	89091	4
57	77896	90263	78886	89683	79840	89081	3
58	77913	90254	78902	89673	79856	89071	2
59	77930	90244	78918	89663	79872	89060	1
60	77946	90235	78934	89653	79887	89050	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	53°		52°		51°		

′	39° Sine	39° Cosine	40° Sine	40° Cosine	41° Sine	41° Cosine	′
0	9.79887	9.89050	9.80807	9.88425	9.81694	9.87778	60
1	79903	89040	80822	88415	81709	87767	59
2	79918	89030	80837	88404	81723	87756	58
3	79934	89020	80852	88394	81738	87745	57
4	79950	89009	80867	88383	81752	87734	56
5	79965	88999	80882	88372	81767	87723	55
6	79981	88989	80897	88362	81781	87712	54
7	79996	88978	80912	88351	81796	87701	53
8	80012	88968	80927	88340	81810	87690	52
9	80027	88958	80942	88330	81825	87679	51
10	9.80043	9.88948	9.80957	9.88319	9.81839	9.87668	50
11	80058	88937	80972	88308	81854	87657	49
12	80074	88927	80987	88298	81868	87646	48
13	80089	88917	81002	88287	81882	87635	47
14	80105	88906	81017	88276	81897	87624	46
15	80120	88896	81032	88266	81911	87613	45
16	80136	88886	81047	88255	81926	87601	44
17	80151	88875	81061	88244	81940	87590	43
18	80166	88865	81076	88234	81955	87579	42
19	80182	88855	81091	88223	81969	87568	41
20	9.80197	9.88844	9.81106	9.88212	9.81983	9.87557	40
21	80213	88834	81121	88201	81998	87546	39
22	80228	88824	81136	88191	82012	87535	38
23	80244	88813	81151	88180	82026	87524	37
24	80259	88803	81166	88169	82041	87513	36
25	80274	88793	81180	88158	82055	87501	35
26	80290	88782	81195	88148	82069	87490	34
27	80305	88772	81210	88137	82084	87479	33
28	80320	88761	81225	88126	82098	87468	32
29	80336	88751	81240	88115	82112	87457	31
30	9.80351	9.88741	9.81254	9.88105	9.82126	9.87446	30
31	80366	88730	81269	88094	82141	87434	29
32	80382	88720	81284	88083	82155	87423	28
33	80397	88709	81299	88072	82169	87412	27
34	80412	88699	81314	88061	82184	87401	26
35	80428	88688	81328	88051	82198	87390	25
36	80443	88678	81343	88040	82212	87378	24
37	80458	88668	81358	88029	82226	87367	23
38	80473	88657	81372	88018	82240	87356	22
39	80489	88647	81387	88007	82255	87345	21
40	9.80504	9.88636	9.81402	9.87996	9.82269	9.87334	20
41	80519	88626	81417	87985	82283	87322	19
42	80534	88615	81431	87975	82297	87311	18
43	80550	88605	81446	87964	82311	87300	17
44	80565	88594	81461	87953	82326	87288	16
45	80580	88584	81475	87942	82340	87277	15
46	80595	88573	81490	87931	82354	87266	14
47	80610	88563	81505	87920	82368	87255	13
48	80625	88552	81519	87909	82382	87243	12
49	80641	88542	81534	87898	82396	87232	11
50	9.80656	9.88531	9.81549	9.87887	9.82410	9.87221	10
51	80671	88521	81563	87877	82424	87209	9
52	80686	88510	81578	87866	82439	87198	8
53	80701	88499	81592	87855	82453	87187	7
54	80716	88489	81607	87844	82467	87175	6
55	80731	88478	81622	87833	82481	87164	5
56	80746	88468	81636	87822	82495	87153	4
57	80762	88457	81651	87811	82509	87141	3
58	80777	88447	81665	87800	82523	87130	2
59	80792	88436	81680	87789	82537	87119	1
60	80807	88425	81694	87778	82551	87107	0
′	50° Cosine	50° Sine	49° Cosine	49° Sine	48° Cosine	48° Sine	′

′	42°		43°		44°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	9.82551	9.87107	9.83378	9.86413	9.84177	9.85693	**60**
1	82565	87096	83392	86401	84190	85681	**59**
2	82579	87085	83405	86389	84203	85669	**58**
3	82593	87073	83419	86377	84216	85657	**57**
4	82607	87062	83432	86366	84229	85645	**56**
5	82621	87050	83446	86354	84242	85632	**55**
6	82635	87039	83459	86342	84255	85620	**54**
7	82649	87028	83473	86330	84269	85608	**53**
8	82663	87016	83486	86318	84282	85596	**52**
9	82677	87005	83500	86306	84295	85583	**51**
10	9.82691	9.86993	9.83513	9.86295	9.84308	9.85571	**50**
11	82705	86982	83527	86283	84321	85559	**49**
12	82719	86970	83540	86271	84334	85547	**48**
13	82733	86959	83554	86259	84347	85534	**47**
14	82747	86947	83567	86247	84360	85522	**46**
15	82761	86936	83581	86235	84373	85510	**45**
16	82775	86924	83594	86223	84385	85497	**44**
17	82788	86913	83608	86211	84398	85485	**43**
18	82802	86902	83621	86200	84411	85473	**42**
19	82816	86890	83634	86188	84424	85460	**41**
20	9.82830	9.86879	9.83648	9.86176	9.84437	9.85448	**40**
21	82844	86867	83661	86164	84450	85436	**39**
22	82858	86855	83674	86152	84463	85423	**38**
23	82872	86844	83688	86140	84476	85411	**37**
24	82885	86832	83701	86128	84489	85399	**36**
25	82899	86821	83715	86116	84502	85386	**35**
26	82913	86809	83728	86104	84515	85374	**34**
27	82927	86798	83741	86092	84528	85361	**33**
28	82941	86786	83755	86080	84540	85349	**32**
29	82955	86775	83768	86068	84553	85337	**31**
30	9.82968	9.86763	9.83781	9.86056	9.84566	9.85324	**30**
31	82982	86752	83795	86044	84579	85312	**29**
32	82996	86740	83808	86032	84592	85299	**28**
33	83010	86728	83821	86020	84605	85287	**27**
34	83023	86717	83834	86008	84618	85274	**26**
35	83037	86705	83848	85996	84630	85262	**25**
36	83051	86694	83861	85984	84643	85250	**24**
37	83065	86682	83874	85972	84656	85237	**23**
38	83078	86670	83887	85960	84669	85225	**22**
39	83092	86659	83901	85948	84682	85212	**21**
40	9.83106	9.86647	9.83914	9.85936	9.84694	9.85200	**20**
41	83120	86635	83927	85924	84707	85187	**19**
42	83133	86624	83940	85912	84720	85175	**18**
43	83147	86612	83954	85900	84733	85162	**17**
44	83161	86600	83967	85888	84745	85150	**16**
45	83174	86589	83980	85876	84758	85137	**15**
46	83188	86577	83993	85864	84771	85125	**14**
47	83202	86565	84006	85851	84784	85112	**13**
48	83215	86554	84020	85839	84796	85100	**12**
49	83229	86542	84033	85827	84809	85087	**11**
50	9.83242	9.86530	9.84046	9.85815	9.84822	9.85074	**10**
51	83256	86518	84059	85803	84835	85062	**9**
52	83270	86507	84072	85791	84847	85049	**8**
53	83283	86495	84085	85779	84860	85037	**7**
54	83297	86483	84098	85766	84873	85024	**6**
55	83310	86472	84112	85754	84885	85012	**5**
56	83324	86460	84125	85742	84898	84999	**4**
57	83338	86448	84138	85730	84911	84986	**3**
58	83351	86436	84151	85718	84923	84974	**2**
59	83365	86425	84164	85706	84936	84961	**1**
60	83378	86413	84177	85693	84949	84949	**0**
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	47°		46°		45°		

′	0°		1°		2°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	− ∞	∞	8.24192	11.75808	8.54308	11.45692	60
1	6.46373	13.53627	24910	75090	54669	45331	59
2	76476	23524	25616	74384	55027	44973	58
3	94085	05915	26312	73688	55382	44618	57
4	7.06579	12.93421	26996	73004	55734	44266	56
5	16270	83730	27669	72331	56083	43917	55
6	24188	75812	28332	71668	56429	43571	54
7	30882	69118	28986	71014	56773	43227	53
8	36682	63318	29629	70371	57114	42886	52
9	41797	58203	30263	69737	57452	42548	51
10	7.46373	12.53627	8.30888	11.69112	8.57788	11.42212	50
11	50512	49488	31505	68495	58121	41879	49
12	54291	45709	32112	67888	58451	41549	48
13	57767	42233	32711	67289	58779	41221	47
14	60986	39014	33302	66698	59105	40895	46
15	63982	36018	33886	66114	59428	40572	45
16	66785	33215	34461	65539	59749	40251	44
17	69418	30582	35029	64971	60068	39932	43
18	71900	28100	35590	64410	60384	39616	42
19	74248	25752	36143	63857	60698	39302	41
20	7.76476	12.23524	8.36689	11.63311	8.61009	11.38991	40
21	78595	21405	37229	62771	61319	38681	39
22	80615	19385	37762	62238	61626	38374	38
23	82546	17454	38289	61711	61931	38069	37
24	84394	15606	38809	61191	62234	37766	36
25	86167	13833	39323	60677	62535	37465	35
26	87871	12129	39832	60168	62834	37166	34
27	89510	10490	40334	59666	63131	36869	33
28	91089	08911	40830	59170	63426	36574	32
29	92613	07387	41321	58679	63718	36282	31
30	7.94086	12.05914	8.41807	11.58193	8.64009	11.35991	30
31	95510	04490	42287	57713	64298	35702	29
32	96889	03111	42762	57238	64585	35415	28
33	98225	01775	43232	56768	64870	35130	27
34	99522	00478	43696	56304	65154	34846	26
35	8.00781	11.99219	44156	55844	65435	34565	25
36	02004	97996	44611	55389	65715	34285	24
37	03194	96806	45061	54939	65993	34007	23
38	04353	95647	45507	54493	66269	33731	22
39	05481	94519	45948	54052	66543	33457	21
40	8.06581	11.93419	8.46385	11.53615	8.66816	11.33184	20
41	07653	92347	46817	53183	67087	32913	19
42	08700	91300	47245	52755	67356	32644	18
43	09722	90278	47669	52331	67624	32376	17
44	10720	89280	48089	51911	67890	32110	16
45	11696	88304	48505	51495	68154	31846	15
46	12651	87349	48917	51083	68417	31583	14
47	13585	86415	49325	50675	68678	31322	13
48	14500	85500	49729	50271	68938	31062	12
49	15395	84605	50130	49870	69196	30804	11
50	8.16273	11.83727	8.50527	11.49473	8.69453	11.30547	10
51	17133	82867	50920	49080	69708	30292	9
52	17976	82024	51310	48690	69962	30038	8
53	18804	81196	51696	48304	70214	29786	7
54	19616	80384	52079	47921	70465	29535	6
55	20413	79587	52459	47541	70714	29286	5
56	21195	78805	52835	47165	70962	29038	4
57	21964	78036	53208	46792	71208	28792	3
58	22720	77280	53578	46422	71453	28547	2
59	23462	76538	53945	46055	71697	28303	1
60	24192	75808	54308	45692	71940	28060	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	89°		88°		87°		

′	3°		4°		5°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	8.71940	11.28060	8.84464	11.15536	8.94195	11.05805	60
1	72181	27819	84646	15354	94340	05660	59
2	72420	27580	84826	15174	94485	05515	58
3	72659	27341	85006	14994	94630	05370	57
4	72896	27104	85185	14815	94773	05227	56
5	73132	26868	85363	14637	94917	05083	55
6	73366	26634	85540	14460	95060	04940	54
7	73600	26400	85717	14283	95202	04798	53
8	73832	26168	85893	14107	95344	04656	52
9	74063	25937	86069	13931	95486	04514	51
10	8.74292	11.25708	8.86243	11.13757	8.95627	11.04373	50
11	74521	25479	86417	13583	95767	04233	49
12	74748	25252	86591	13409	95908	04092	48
13	74974	25026	86763	13237	96047	03953	47
14	75199	24801	86935	13065	96187	03813	46
15	75423	24577	87106	12894	96325	03675	45
16	75645	24355	87277	12723	96464	03536	44
17	75867	24133	87447	12553	96602	03398	43
18	76087	23913	87616	12384	96739	03261	42
19	76306	23694	87785	12215	96877	02123	41
20	8.76525	11.23475	8.87953	11.12047	8.97013	11.02987	40
21	76742	23258	88120	11880	97150	02850	39
22	76958	23042	88287	11713	97285	02715	38
23	77173	22827	88453	11547	97421	02579	37
24	77387	22613	88618	11382	97556	02444	36
25	77600	22400	88783	11217	97691	02309	35
26	77811	22189	88948	11052	97825	02175	34
27	78022	21978	89111	10889	97959	02041	33
28	78232	21768	89274	10726	98092	01908	32
29	78441	21559	89437	10563	98225	01775	31
30	8.78649	11.21351	8.89598	11.10402	8.98358	11.01642	30
31	78855	21145	89760	10240	98490	01510	29
32	79061	20939	89920	10080	98622	01378	28
33	79266	20734	90080	09920	98753	01247	27
34	79470	20530	90240	09760	98884	01116	26
35	79673	20327	90399	09601	99015	00985	25
36	79875	20125	90557	09443	99145	00855	24
37	80076	19924	90715	09285	99275	00725	23
38	80277	19723	90872	09128	99405	00595	22
39	80476	19524	91029	08971	99534	00466	21
40	8.80674	11.19326	8.91185	11.08815	8.99662	11.00338	20
41	80872	19128	91340	08660	99791	00209	19
42	81068	18932	91495	08505	99919	00081	18
43	81264	18736	91650	08350	9.00046	10.99954	17
44	81459	18541	91803	08197	00174	99826	16
45	81653	18347	91957	08043	00301	99699	15
46	81846	18154	92110	07890	00427	99573	14
47	82038	17962	92262	07738	00553	99447	13
48	82230	17770	92414	07586	00679	99321	12
49	82420	17580	92565	07435	00805	99195	11
50	8.82610	11.17390	8.92716	11.07284	9.00930	10.99070	10
51	82799	17201	92866	07134	01055	98945	9
52	82987	17013	93016	06984	01179	98821	8
53	83175	16825	93165	06835	01303	98697	7
54	83361	16639	93313	06687	01427	98573	6
55	83547	16453	93462	06538	01550	98450	5
56	83732	16268	93609	06391	01673	98327	4
57	83916	16084	93756	06244	01796	98204	3
58	84100	15900	93903	06097	01918	98082	2
59	84282	15718	94049	05951	02040	97960	1
60	84464	15536	94195	05805	02162	97838	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	86°		85°		84°		

′	6°		7°		8°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	9.02162	10.97838	9.08914	10.91086	9.14780	10.85220	60
1	02283	97717	09019	90981	14872	85128	59
2	02404	97596	09123	90877	14963	85037	58
3	02525	97475	09227	90773	15054	84946	57
4	02645	97355	09330	90670	15145	84855	56
5	02766	97234	09434	90566	15236	84764	55
6	02885	97115	09537	90463	15327	84673	54
7	03005	96995	09640	90360	15417	84583	53
8	03124	96876	09742	90258	15508	84492	52
9	03242	96758	09845	90155	15598	84402	51
10	9.03361	10.96639	9.09947	10.90053	9.15688	10.84312	50
11	03479	96521	10049	89951	15777	84223	49
12	03597	96403	10150	89850	15867	84133	48
13	03714	96286	10252	89748	15956	84044	47
14	03832	96168	10353	89647	16046	83954	46
15	03948	96052	10454	89546	16135	83865	45
16	04065	95935	10555	89445	16224	83776	44
17	04181	95819	10656	89344	16312	83688	43
18	04297	95703	10756	89244	16401	83599	42
19	04413	95587	10856	89144	16489	83511	41
20	9.04528	10.95472	9.10956	10.89044	9.16577	10.83423	40
21	04643	95357	11056	88944	16665	83335	39
22	04758	95242	11155	88845	16753	83247	38
23	04873	95127	11254	88746	16841	83159	37
24	04987	95013	11353	88647	16928	83072	36
25	05101	94899	11452	88548	17016	82984	35
26	05214	94786	11551	88449	17103	82897	34
27	05328	94672	11649	88351	17190	82810	33
28	05441	94559	11747	88253	17277	82723	32
29	05553	94447	11845	88155	17363	82637	31
30	9.05666	10.94334	9.11943	10.88057	9.17450	10.82550	30
31	05778	94222	12040	87960	17536	82464	29
32	05890	94110	12138	87862	17622	82378	28
33	06002	93998	12235	87765	17708	82292	27
34	06113	93887	12332	87668	17794	82206	26
35	06224	93776	12428	87572	17880	82120	25
36	06335	93665	12525	87475	17965	82035	24
37	06445	93555	12621	87379	18051	81949	23
38	06556	93444	12717	87283	18136	81864	22
39	06666	93334	12813	87187	18221	81779	21
40	9.06775	10.93225	9.12909	10.87091	9.18306	10.81694	20
41	06885	93115	13004	86996	18391	81609	19
42	06994	93006	13099	86901	18475	81525	18
43	07103	92897	13194	86806	18560	81440	17
44	07211	92789	13289	86711	18644	81356	16
45	07320	92680	13384	86616	18728	81272	15
46	07428	92572	13478	86522	18812	81188	14
47	07536	92464	13573	86427	18896	81104	13
48	07643	92357	13667	86333	18979	81021	12
49	07751	92249	13761	86239	19063	80937	11
50	9.07858	10.92142	9.13854	10.86146	9.19146	10.80854	10
51	07964	92036	13948	86052	19229	80771	9
52	08071	91929	14041	85959	19312	80688	8
53	08177	91823	14134	85866	19395	80605	7
54	08283	91717	14227	85773	19478	80522	6
55	08389	91611	14320	85680	19561	80439	5
56	08495	91505	14412	85588	19643	80357	4
57	08600	91400	14504	85496	19725	80275	3
58	08705	91295	14597	85403	19807	80193	2
59	08810	91190	14688	85312	19889	80111	1
60	08914	91086	14780	85220	19971	80029	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	83°		82°		81°		

′	9°		10°		11°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	9.19971	10.80029	9.24632	10.75368	9.28865	10.71135	60
1	20053	79947	24706	75294	28933	71067	59
2	20134	79866	24779	75221	29000	71000	58
3	20216	79784	24853	75147	29067	70933	57
4	20297	79703	24926	75074	29134	70866	56
5	20378	79622	25000	75000	29201	70799	55
6	20459	79541	25073	74927	29268	70732	54
7	20540	79460	25146	74854	29335	70665	53
8	20621	79379	25219	74781	29402	70598	52
9	20701	79299	25292	74708	29468	70532	51
10	9.20782	10.79218	9.25365	10.74635	9.29535	10.70465	50
11	20862	79138	25437	74563	29601	70399	49
12	20942	79058	25510	74490	29668	70332	48
13	21022	78978	25582	74418	29734	70266	47
14	21102	78898	25655	74345	29800	70200	46
15	21182	78818	25727	74273	29866	70134	45
16	21261	78739	25799	74201	29932	70068	44
17	21341	78659	25871	74129	29998	70002	43
18	21420	78580	25943	74057	30064	69936	42
19	21499	78501	26015	73985	30130	69870	41
20	9.21578	10.78422	9.26086	10.73914	9.30195	10.69805	40
21	21657	78343	26158	73842	30261	69739	39
22	21736	78264	26229	73771	30326	69674	38
23	21814	78186	26301	73699	30391	69609	37
24	21893	78107	26372	73628	30457	69543	36
25	21971	78029	26443	73557	30522	69478	35
26	22049	77951	26514	73486	30587	69413	34
27	22127	77873	26585	73415	30652	69348	33
28	22205	77795	26655	73345	30717	69283	32
29	22283	77717	26726	73274	30782	69218	31
30	9.22361	10.77639	9.26797	10.73203	9.30846	10.69154	30
31	22438	77562	26867	73133	30911	69089	29
32	22516	77484	26937	73063	30975	69025	28
33	22593	77407	27008	72992	31040	68960	27
34	22670	77330	27078	72922	31104	68896	26
35	22747	77253	27148	72852	31168	68832	25
36	22824	77176	27218	72782	31233	68767	24
37	22901	77099	27288	72712	31297	68703	23
38	22977	77023	27357	72643	31361	68639	22
39	23054	76946	27427	72573	31425	68575	21
40	9.23130	10.76870	9.27496	10.72504	9.31489	10.68511	20
41	23206	76794	27566	72434	31552	68448	19
42	23283	76717	27635	72365	31616	68384	18
43	23359	76641	27704	72296	31679	68321	17
44	23435	76565	27773	72227	31743	68257	16
45	23510	76490	27842	72158	31806	68194	15
46	23586	76414	27911	72089	31870	68130	14
47	23661	76339	27980	72020	31933	68067	13
48	23737	76263	28049	71951	31996	68004	12
49	23812	76188	28117	71883	32059	67941	11
50	9.23887	10.76113	9.28186	10.71814	9.32122	10.67878	10
51	23962	76038	28254	71746	32185	67815	9
52	24037	75963	28323	71677	32248	67752	8
53	24112	75888	28391	71609	32311	67689	7
54	24186	75814	28459	71541	32373	67627	6
55	24261	75739	28527	71473	32436	67564	5
56	24335	75665	28595	71405	32498	67502	4
57	24410	75590	28662	71338	32561	67439	3
58	24484	75516	28730	71270	32623	67377	2
59	24558	75442	28798	71202	32685	67315	1
60	24632	75368	28865	71135	32747	67253	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	80°		79°		78°		

′	12°		13°		14°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	9.32747	10.67253	9.36336	10.63664	9.39677	10.60323	60
1	32810	67190	36394	63606	39731	60269	59
2	32872	67128	36452	63548	39785	60215	58
3	32933	67067	36509	63491	39838	60162	57
4	32995	67005	36566	63434	39892	60108	56
5	33057	66943	36624	63376	39945	60055	55
6	33119	66881	36681	63319	39999	60001	54
7	33180	66820	36738	63262	40052	59948	53
8	33242	66758	36795	63205	40106	59894	52
9	33303	66697	36852	63148	40159	59841	51
10	9.33365	10.66635	9.36909	10.63091	9.40212	10.59788	50
11	33426	66574	36966	63034	40266	59734	49
12	33487	66513	37023	62977	40319	59681	48
13	33548	66452	37080	62920	40372	59628	47
14	33609	66391	37137	62863	40425	59575	46
15	33670	66330	37193	62807	40478	59522	45
16	33731	66269	37250	62750	40531	59469	44
17	33792	66208	37306	62694	40584	59416	43
18	33853	66147	37363	62637	40636	59364	42
19	33913	66087	37419	62581	40689	59311	41
20	9.33974	10.66026	9.37476	10.62524	9.40742	10.59258	40
21	34034	65966	37532	62468	40795	59205	39
22	34095	65905	37588	62412	40847	59153	38
23	34155	65845	37644	62356	40900	59100	37
24	34215	65785	37700	62300	40952	59048	36
25	34276	65724	37756	62244	41005	58995	35
26	34336	65664	37812	62188	41057	58943	34
27	34396	65604	37868	62132	41109	58891	33
28	34456	65544	37924	62076	41161	58839	32
29	34516	65484	37980	62020	41214	58786	31
30	9.34576	10.65424	9.38035	10.61965	9.41266	10.58734	30
31	34635	65365	38091	61909	41318	58682	29
32	34695	65305	38147	61853	41370	58630	28
33	34755	65245	38202	61798	41422	58578	27
34	34814	65186	38257	61743	41474	58526	26
35	34874	65126	38313	61687	41526	58474	25
36	34933	65067	38368	61632	41578	58422	24
37	34992	65008	38423	61577	41629	58371	23
38	35051	64949	38479	61521	41681	58319	22
39	35111	64889	38534	61466	41733	58267	21
40	9.35170	10.64830	9.38589	10.61411	9.41784	10.58216	20
41	35229	64771	38644	61356	41836	58164	19
42	35288	64712	38699	61301	41887	58113	18
43	35347	64653	38754	61246	41939	58061	17
44	35405	64595	38808	61192	41990	58010	16
45	35464	64536	38863	61137	42041	57959	15
46	35523	64477	38918	61082	42093	57907	14
47	35581	64419	38972	61028	42144	57856	13
48	35640	64360	39027	60973	42195	57805	12
49	35698	64302	39082	60918	42246	57754	11
50	9.35757	10.64243	9.39136	10.60864	9.42297	10.57703	10
51	35815	64185	39190	60810	42348	57652	9
52	35873	64127	39245	60755	42399	57601	8
53	35931	64069	39299	60701	42450	57550	7
54	35989	64011	39353	60647	42501	57499	6
55	36047	63953	39407	60593	42552	57448	5
56	36105	63895	39461	60539	42603	57397	4
57	36163	63837	39515	60485	42653	57347	3
58	36221	63779	39569	60431	42704	57296	2
59	36279	63721	39623	60377	42755	57245	1
60	36336	63664	39677	60323	42805	57195	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	77°		76°		75°		

′	15° Tan	15° Cotan	16° Tan	16° Cotan	17° Tan	17° Cotan	′
0	9.42805	10.57195	9.45750	10.54250	9.48534	10.51466	60
1	42856	57144	45797	54203	48579	51421	59
2	42906	57094	45845	54155	48624	51376	58
3	42957	57043	45892	54108	48669	51331	57
4	43007	56993	45940	54060	48714	51286	56
5	43057	56943	45987	54013	48759	51241	55
6	43108	56892	46035	53965	48804	51196	54
7	43158	56842	46082	53918	48849	51151	53
8	43208	56792	46130	53870	48894	51106	52
9	43258	56742	46177	53823	48939	51061	51
10	9.43308	10.56692	9.46224	10.53776	9.48984	10.51016	50
11	43358	56642	46271	53729	49029	50971	49
12	43408	56592	46319	53681	49073	50927	48
13	43458	56542	46366	53634	49118	50882	47
14	43508	56492	46413	53587	49163	50837	46
15	43558	56442	46460	53540	49207	50793	45
16	43607	56393	46507	53493	49252	50748	44
17	43657	56343	46554	53446	49296	50704	43
18	43707	56293	46601	53399	49341	50659	42
19	43756	56244	46648	53352	49385	50615	41
20	9.43806	10.56194	9.46694	10.53306	9.49430	10.50570	40
21	43855	56145	46741	53259	49474	50526	39
22	43905	56095	46788	53212	49519	50481	38
23	43954	56046	46835	53165	49563	50437	37
24	44004	55996	46881	53119	49607	50393	36
25	44053	55947	46928	53072	49652	50348	35
26	44102	55898	46975	53025	49696	50304	34
27	44151	55849	47021	52979	49740	50260	33
28	44201	55799	47068	52932	49784	50216	32
29	44250	55750	47114	52886	49828	50172	31
30	9.44299	10.55701	9.47160	10.52840	9.49872	10.50128	30
31	44348	55652	47207	52793	49916	50084	29
32	44397	55603	47253	52747	49960	50040	28
33	44446	55554	47299	52701	50004	49996	27
34	44495	55505	47346	52654	50048	49952	26
35	44544	55456	47392	52608	50092	49908	25
36	44592	55408	47438	52562	50136	49864	24
37	44641	55359	47484	52516	50180	49820	23
38	44690	55310	47530	52470	50223	49777	22
39	44738	55262	47576	52424	50267	49733	21
40	9.44787	10.55213	9.47622	10.52378	9.50311	10.49689	20
41	44836	55164	47668	52332	50355	49645	19
42	44884	55116	47714	52286	50398	49602	18
43	44933	55067	47760	52240	50442	49558	17
44	44981	55019	47806	52194	50485	49515	16
45	45029	54971	47852	52148	50529	49471	15
46	45078	54922	47897	52103	50572	49428	14
47	45126	54874	47943	52057	50616	49384	13
48	45174	54826	47989	52011	50659	49341	12
49	45222	54778	48035	51965	50703	49297	11
50	9.45271	10.54729	9.48080	10.51920	9.50746	10.49254	10
51	45319	54681	48126	51874	50789	49211	9
52	45367	54633	48171	51829	50833	49167	8
53	45415	54585	48217	51783	50876	49124	7
54	45463	54537	48262	51738	50919	49081	6
55	45511	54489	48307	51693	50962	49038	5
56	45559	54441	48353	51647	51005	48995	4
57	45606	54394	48398	51602	51048	48952	3
58	45654	54346	48443	51557	51092	48908	2
59	45702	54298	48489	51511	51135	48865	1
60	45750	54250	48534	51466	51178	48822	0
′	74° Cotan	74° Tan	73° Cotan	73° Tan	72° Cotan	72° Tan	′

′	18°		19°		20°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	9.51178	10.48822	9.53697	10.46303	9.56107	10.43893	60
1	51221	48779	53738	46262	56146	43854	59
2	51264	48736	53779	46221	56185	43815	58
3	51306	48694	53820	46180	56224	43776	57
4	51349	48651	53861	46139	56264	43736	56
5	51392	48608	53902	46098	56303	43697	55
6	51435	48565	53943	46057	56342	43658	54
7	51478	48522	53984	46016	56381	43619	53
8	51520	48480	54025	45975	56420	43580	52
9	51563	48437	54065	45935	56459	43541	51
10	9.51606	10.48394	9.54106	10.45894	9.56498	10.43502	50
11	51648	48352	54147	45853	56537	43463	49
12	51691	48309	54187	45813	56576	43424	48
13	51734	48266	54228	45772	56615	43385	47
14	51776	48224	54269	45731	56654	43346	46
15	51819	48181	54309	45691	56693	43307	45
16	51861	48139	54350	45650	56732	43268	44
17	51903	48097	54390	45610	56771	43229	43
18	51946	48054	54431	45569	56810	43190	42
19	51988	48012	54471	45529	56849	43151	41
20	9.52031	10.47969	9.54512	10.45488	9.56887	10.43113	40
21	52073	47927	54552	45448	56926	43074	39
22	52115	47885	54593	45407	56965	43035	38
23	52157	47843	54633	45367	57004	42996	37
24	52200	47800	54673	45327	57042	42958	36
25	52242	47758	54714	45286	57081	42919	35
26	52284	47716	54754	45246	57120	42880	34
27	52326	47674	54794	45206	57158	42842	33
28	52368	47632	54835	45165	57197	42803	32
29	52410	47590	54875	45125	57235	42765	31
30	9.52452	10.47548	9.54915	10.45085	9.57274	10.42726	30
31	52494	47506	54955	45045	57312	42688	29
32	52536	47464	54995	45005	57351	42649	28
33	52578	47422	55035	44965	57389	42611	27
34	52620	47380	55075	44925	57428	42572	26
35	52661	47339	55115	44885	57466	42534	25
36	52703	47297	55155	44845	57504	42496	24
37	52745	47255	55195	44805	57543	42457	23
38	52787	47213	55235	44765	57581	42419	22
39	52829	47171	55275	44725	57619	42381	21
40	9.52870	10.47130	9.55315	10.44685	9.57658	10.42342	20
41	52912	47088	55355	44645	57696	42304	19
42	52953	47047	55395	44605	57734	42266	18
43	52995	47005	55434	44566	57772	42228	17
44	53037	46963	55474	44526	57810	42190	16
45	53078	46922	55514	44486	57849	42151	15
46	53120	46880	55554	44446	57887	42113	14
47	53161	46839	55593	44407	57925	42075	13
48	53202	46798	55633	44367	57963	42037	12
49	53244	46756	55673	44327	58001	41999	11
50	9.53285	10.46715	9.55712	10.44288	9.58039	10.41961	10
51	53327	46673	55752	44248	58077	41923	9
52	53368	46632	55791	44209	58115	41885	8
53	53409	46591	55831	44169	58153	41847	7
54	53450	46550	55870	44130	58191	41809	6
55	53492	46508	55910	44090	58229	41771	5
56	53533	46467	55949	44051	58267	41733	4
57	53574	46426	55989	44011	58304	41696	3
58	53615	46385	56028	43972	58342	41658	2
59	53656	46344	56067	43933	58380	41620	1
60	53697	46303	56107	43893	58418	41582	0
	Cotan	Tan	Cotan	Tan	Cotan	Tan	
′	71°		70°		69°		′

′	21°		22°		23°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	9.58418	10.41582	9.60641	10.39359	9.62785	10.37215	60
1	58455	41545	60677	39323	62820	37180	59
2	58493	41507	60714	39286	62855	37145	58
3	58531	41469	60750	39250	62890	37110	57
4	58569	41431	60786	39214	62926	37074	56
5	58606	41394	60823	39177	62961	37039	55
6	58644	41356	60859	39141	62996	37004	54
7	58681	41319	60895	39105	63031	36969	53
8	58719	41281	60931	39069	63066	36934	52
9	58757	41243	60967	39033	63101	36899	51
10	9.58794	10.41206	9.61004	10.38996	9.63135	10.36865	50
11	58832	41168	61040	38960	63170	36830	49
12	58869	41131	61076	38924	63205	36795	48
13	58907	41093	61112	38888	63240	36760	47
14	58944	41056	61148	38852	63275	36725	46
15	58981	41019	61184	38816	63310	36690	45
16	59019	40981	61220	38780	63345	36655	44
17	59056	40944	61256	38744	63379	36621	43
18	59094	40906	61292	38708	63414	36586	42
19	59131	40869	61328	38672	63449	36551	41
20	9.59168	10.40832	9.61364	10.38636	9.63484	10.36516	40
21	59205	40795	61400	38600	63519	36481	39
22	59243	40757	61436	38564	63553	36447	38
23	59280	40720	61472	38528	63588	36412	37
24	59317	40683	61508	38492	63623	36377	36
25	59354	40646	61544	38456	63657	36343	35
26	59391	40609	61579	38421	63692	36308	34
27	59429	40571	61615	38385	63726	36274	33
28	59466	40534	61651	38349	63761	36239	32
29	59503	40497	61687	38313	63796	36204	31
30	9.59540	10.40460	9.61722	10.38278	9.63830	10.36170	30
31	59577	40423	61758	38242	63865	36135	29
32	59614	40386	61794	38206	63899	36101	28
33	59651	40349	61830	38170	63934	36066	27
34	59688	40312	61865	38135	63968	36032	26
35	59725	40275	61901	38099	64003	35997	25
36	59762	40238	61936	38064	64037	35963	24
37	59799	40201	61972	38028	64072	35928	23
38	59835	40165	62008	37992	64106	35894	22
39	59872	40128	62043	37957	64140	35860	21
40	9.59909	10.40091	9.62079	10.37921	9.64175	10.35825	20
41	59946	40054	62114	37886	64209	35791	19
42	59983	40017	62150	37850	64243	35757	18
43	60019	39981	62185	37815	64278	35722	17
44	60056	39944	62221	37779	64312	35688	16
45	60093	39907	62256	37744	64346	35654	15
46	60130	39870	62292	37708	64381	35619	14
47	60166	39834	62327	37673	64415	35585	13
48	60203	39797	62362	37638	64449	35551	12
49	60240	39760	62398	37602	64483	35517	11
50	9.60276	10.39724	9.62433	10.37567	9.64517	10.35483	10
51	60313	39687	62468	37532	64552	35448	9
52	60349	39651	62504	37496	64586	35414	8
53	60386	39614	62539	37461	64620	35380	7
54	60422	39578	62574	37426	64654	35346	6
55	60459	39541	62609	37391	64688	35312	5
56	60495	39505	62645	37355	64722	35278	4
57	60532	39468	62680	37320	64756	35244	3
58	60568	39432	62715	37285	64790	35210	2
59	60605	39395	62750	37250	64824	35176	1
60	60641	39359	62785	37215	64858	35142	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	68°		67°		66°		

′	24°		25°		26°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	9.64858	10.35142	9.66867	10.33133	9.68818	10.31182	60
1	64892	35108	66900	33100	68850	31150	59
2	64926	35074	66933	33067	68882	31118	58
3	64960	35040	66966	33034	68914	31086	57
4	64994	35006	66999	33001	68946	31054	56
5	65028	34972	67032	32968	68978	31022	55
6	65062	34938	67065	32935	69010	30990	54
7	65096	34904	67098	32902	69042	30958	53
8	65130	34870	67131	32869	69074	30926	52
9	65164	34836	67163	32837	69106	30894	51
10	9.65197	10.34803	9.67196	10.32804	9.69138	10.30862	50
11	65231	34769	67229	32771	69170	30830	49
12	65265	34735	67262	32738	69202	30798	48
13	65299	34701	67295	32705	69234	30766	47
14	65333	34667	67327	32673	69266	30734	46
15	65366	34634	67360	32640	69298	30702	45
16	65400	34600	67393	32607	69329	30671	44
17	65434	34566	67426	32574	69361	30639	43
18	65467	34533	67458	32542	69393	30607	42
19	65501	34499	67491	32509	69425	30575	41
20	9.65535	10.34465	9.67524	10.32476	9.69457	10.30543	40
21	65568	34432	67556	32444	69488	30512	39
22	65602	34398	67589	32411	69520	30480	38
23	65636	34364	67622	32378	69552	30448	37
24	65669	34331	67654	32346	69584	30416	36
25	65703	34297	67687	32313	69615	30385	35
26	65736	34264	67719	32281	69647	30353	34
27	65770	34230	67752	32248	69679	30321	33
28	65803	34197	67785	32215	69710	30290	32
29	65837	34163	67817	32183	69742	30258	31
30	9.65870	10.34130	9.67850	10.32150	9.69774	10.30226	30
31	65904	34096	67882	32118	69805	30195	29
32	65937	34063	67915	32085	69837	30163	28
33	65971	34029	67947	32053	69868	30132	27
34	66004	33996	67980	32020	69900	30100	26
35	66038	33962	68012	31988	69932	30068	25
36	66071	33929	68044	31956	69963	30037	24
37	66104	33896	68077	31923	69995	30005	23
38	66138	33862	68109	31891	70026	29974	22
39	66171	33829	68142	31858	70058	29942	21
40	9.66204	10.33796	9.68174	10.31826	9.70089	10.29911	20
41	66238	33762	68206	31794	70121	29879	19
42	66271	33729	68239	31761	70152	29848	18
43	66304	33696	68271	31729	70184	29816	17
44	66337	33663	68303	31697	70215	29785	16
45	66371	33629	68336	31664	70247	29753	15
46	66404	33596	68368	31632	70278	29722	14
47	66437	33563	68400	31600	70309	29691	13
48	66470	33530	68432	31568	70341	29659	12
49	66503	33497	68465	31535	70372	29628	11
50	9.66537	10.33463	9.68497	10.31503	9.70404	10.29596	10
51	66570	33430	68529	31471	70435	29565	9
52	66603	33397	68561	31439	70466	29534	8
53	66636	33364	68593	31407	70498	29502	7
54	66669	33331	68626	31374	70529	29471	6
55	66702	33298	68658	31342	70560	29440	5
56	66735	33265	68690	31310	70592	29408	4
57	66768	33232	68722	31278	70623	29377	3
58	66801	33199	68754	31246	70654	29346	2
59	66834	33166	68786	31214	70685	29315	1
60	66867	33133	68818	31182	70717	29283	0
	Cotan	Tan	Cotan	Tan	Cotan	Tan	
′	65°		64°		63°		′

′	27° Tan	27° Cotan	28° Tan	28° Cotan	29° Tan	29° Cotan	′
0	9.70717	10.29283	9.72567	10.27433	9.74375	10.25625	60
1	70748	29252	72598	27402	74405	25595	59
2	70779	29221	72628	27372	74435	25565	58
3	70810	29190	72659	27341	74465	25535	57
4	70841	29159	72689	27311	74494	25506	56
5	70873	29127	72720	27280	74524	25476	55
6	70904	29096	72750	27250	74554	25446	54
7	70935	29065	72780	27220	74583	25417	53
8	70966	29034	72811	27189	74613	25387	52
9	70997	29003	72841	27159	74643	25357	51
10	9.71028	10.28972	9.72872	10.27128	9.74673	10.25327	50
11	71059	28941	72902	27098	74702	25298	49
12	71090	28910	72932	27068	74732	25268	48
13	71121	28879	72963	27037	74762	25238	47
14	71153	28847	72993	27007	74791	25209	46
15	71184	28816	73023	26977	74821	25179	45
16	71215	28785	73054	26946	74851	25149	44
17	71246	28754	73084	26916	74880	25120	43
18	71277	28723	73114	26886	74910	25090	42
19	71308	28692	73144	26856	74939	25061	41
20	9.71339	10.28661	9.73175	10.26825	9.74969	10.25031	40
21	71370	28630	73205	26795	74998	25002	39
22	71401	28599	73235	26765	75028	24972	38
23	71431	28569	73265	26735	75058	24942	37
24	71462	28538	73295	26705	75087	24913	36
25	71493	28507	73326	26674	75117	24883	35
26	71524	28476	73356	26644	75146	24854	34
27	71555	28445	73386	26614	75176	24824	33
28	71586	28414	73416	26584	75205	24795	32
29	71617	28383	73446	26554	75235	24765	31
30	9.71648	10.28352	9.73476	10.26524	9.75264	10.24736	30
31	71679	28321	73507	26493	75294	24706	29
32	71709	28291	73537	26463	75323	24677	28
33	71740	28260	73567	26433	75353	24647	27
34	71771	28229	73597	26403	75382	24618	26
35	71802	28198	73627	26373	75411	24589	25
36	71833	28167	73657	26343	75441	24559	24
37	71863	28137	73687	26313	75470	24530	23
38	71894	28106	73717	26283	75500	24500	22
39	71925	28075	73747	26253	75529	24471	21
40	9.71955	10.28045	9.73777	10.26223	9.75558	10.24442	20
41	71986	28014	73807	26193	75588	24412	19
42	72017	27983	73837	26163	75617	24383	18
43	72048	27952	73867	26133	75647	24353	17
44	72078	27922	73897	26103	75676	24324	16
45	72109	27891	73927	26073	75705	24295	15
46	72140	27860	73957	26043	75735	24265	14
47	72170	27830	73987	26013	75764	24236	13
48	72201	27799	74017	25983	75793	24207	12
49	72231	27769	74047	25953	75822	24178	11
50	9.72262	10.27738	9.74077	10.25923	9.75852	10.24148	10
51	72293	27707	74107	25893	75881	24119	9
52	72323	27677	74137	25863	75910	24090	8
53	72354	27646	74166	25834	75939	24061	7
54	72384	27616	74196	25804	75969	24031	6
55	72415	27585	74226	25774	75998	24002	5
56	72445	27555	74256	25744	76027	23973	4
57	72476	27524	74286	25714	76056	23944	3
58	72506	27494	74316	25684	76086	23914	2
59	72537	27463	74345	25655	76115	23885	1
60	72567	27433	74375	25625	76144	23856	0
′	Cotan 62°	Tan 62°	Cotan 61°	Tan 61°	Cotan 60°	Tan 60°	′

TABLE XII LOGARITHMIC TANGENTS AND COTANGENTS *(Continued)*

′	30°		31°		32°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	9.76144	10.23856	9.77877	10.22123	9.79579	10.20421	60
1	76173	23827	77906	22094	79607	20393	59
2	76202	23798	77935	22065	79635	20365	58
3	76231	23769	77963	22037	79663	20337	57
4	76261	23739	77992	22008	79691	20309	56
5	76290	23710	78020	21980	79719	20281	55
6	76319	23681	78049	21951	79747	20253	54
7	76348	23652	78077	21923	79776	20224	53
8	76377	23623	78106	21894	79804	20196	52
9	76406	23594	78135	21865	79832	20168	51
10	9.76435	10.23565	9.78163	10.21837	9.79860	10.20140	50
11	76464	23536	78192	21808	79888	20112	49
12	76493	23507	78220	21780	79916	20084	48
13	76522	23478	78249	21751	79944	20056	47
14	76551	23449	78277	21723	79972	20028	46
15	76580	23420	78306	21694	80000	20000	45
16	76609	23391	78334	21666	80028	19972	44
17	76639	23361	78363	21637	80056	19944	43
18	76668	23332	78391	21609	80084	19916	42
19	76697	23303	78419	21581	80112	19888	41
20	9.76725	10.23275	9.78448	10.21552	9.80140	10.19860	40
21	76754	23246	78476	21524	80168	19832	39
22	76783	23217	78505	21495	80195	19805	38
23	76812	23188	78533	21467	80223	19777	37
24	76841	23159	78562	21438	80251	19749	36
25	76870	23130	78590	21410	80279	19721	35
26	76899	23101	78618	21382	80307	19693	34
27	76928	23072	78647	21353	80335	19665	33
28	76957	23043	78675	21325	80363	19637	32
29	76986	23014	78704	21296	80391	19609	31
30	9.77015	10.22985	9.78732	10.21268	9.80419	10.19581	30
31	77044	22956	78760	21240	80447	19553	29
32	77073	22927	78789	21211	80474	19526	28
33	77101	22899	78817	21183	80502	19498	27
34	77130	22870	78845	21155	80530	19470	26
35	77159	22841	78874	21126	80558	19442	25
36	77188	22812	78902	21098	80586	19414	24
37	77217	22783	78930	21070	80614	19386	23
38	77246	22754	78959	21041	80642	19358	22
39	77274	22726	78987	21013	80669	19331	21
40	9.77303	10.22697	9.79015	10.20985	9.80697	10.19303	20
41	77332	22668	79043	20957	80725	19275	19
42	77361	22639	79072	20928	80753	19247	18
43	77390	22610	79100	20900	80781	19219	17
44	77418	22582	79128	20872	80808	19192	16
45	77447	22553	79156	20844	80836	19164	15
46	77476	22524	79185	20815	80864	19136	14
47	77505	22495	79213	20787	80892	19108	13
48	77533	22467	79241	20759	80919	19081	12
49	77562	22438	79269	20731	80947	19053	11
50	9.77591	10.22409	9.79297	10.20703	9.80975	10.19025	10
51	77619	22381	79326	20674	81003	18997	9
52	77648	22352	79354	20646	81030	18970	8
53	77677	22323	79382	20618	81058	18942	7
54	77706	22294	79410	20590	81086	18914	6
55	77734	22266	79438	20562	81113	18887	5
56	77763	22237	79466	20534	81141	18859	4
57	77791	22209	79495	20505	81169	18831	3
58	77820	22180	79523	20477	81196	18804	2
59	77849	22151	79551	20449	81224	18776	1
60	77877	22123	79579	20421	81252	18748	0
	Cotan	Tan	Cotan	Tan	Cotan	Tan	
′	59°		58°		57°		′

′	33° Tan	33° Cotan	34° Tan	34° Cotan	35° Tan	35° Cotan	′
0	9.81252	10.18748	9.82899	10.17101	9.84523	10.15477	60
1	81279	18721	82926	17074	84550	15450	59
2	81307	18693	82953	17047	84576	15424	58
3	81335	18665	82980	17020	84603	15397	57
4	81362	18638	83008	16992	84630	15370	56
5	81390	18610	83035	16965	84657	15343	55
6	81418	18582	83062	16938	84684	15316	54
7	81445	18555	83089	16911	84711	15289	53
8	81473	18527	83117	16883	84738	15262	52
9	81500	18500	83144	16856	84764	15236	51
10	9.81528	10.18472	9.83171	10.16829	9.84791	10.15209	50
11	81556	18444	83198	16802	84818	15182	49
12	81583	18417	83225	16775	84845	15155	48
13	81611	18389	83252	16748	84872	15128	47
14	81638	18362	83280	16720	84899	15101	46
15	81666	18334	83307	16693	84925	15075	45
16	81693	18307	83334	16666	84952	15048	44
17	81721	18279	83361	16639	84979	15021	43
18	81748	18252	83388	16612	85006	14994	42
19	81776	18224	83415	16585	85033	14967	41
20	9.81803	10.18197	9.83142	10.16558	9.85059	10.14941	40
21	81831	18169	83470	16530	85086	14914	39
22	81858	18142	83497	16503	85113	14887	38
23	81886	18114	83524	16476	85140	14860	37
24	81913	18087	83551	16449	85166	14834	36
25	81941	18059	83578	16422	85193	14807	35
26	81968	18032	83605	16395	85220	14780	34
27	81996	18004	83632	16368	85247	14753	33
28	82023	17977	83659	16341	85273	14727	32
29	82051	17949	83686	16314	85300	14700	31
30	9.82078	10.17922	9.83713	10.16287	9.85327	10.14673	30
31	82106	17894	83740	16260	85354	14646	29
32	82133	17867	83768	16232	85380	14620	28
33	82161	17839	83795	16205	85407	14593	27
34	82188	17812	83822	16178	85434	14566	26
35	82215	17785	83849	16151	85460	14540	25
36	82243	17757	83876	16124	85487	14513	24
37	82270	17730	83903	16097	85514	14486	23
38	82298	17702	83930	16070	85540	14460	22
39	82325	17675	83957	16043	85567	14433	21
40	9.82352	10.17648	9.83984	10.16016	9.85594	10.14406	20
41	82380	17620	84011	15989	85620	14380	19
42	82407	17593	84038	15962	85647	14353	18
43	82435	17565	84065	15935	85674	14326	17
44	82462	17538	84092	15908	85700	14300	16
45	82489	17511	84119	15881	85727	14273	15
46	82517	17483	84146	15854	85754	14246	14
47	82544	17456	84173	15827	85780	14220	13
48	82571	17429	84200	15800	85807	14193	12
49	82599	17401	84227	15773	85834	14166	11
50	9.82626	10.17374	9.84254	10.15746	9.85860	10.14140	10
51	82653	17347	84280	15720	85887	14113	9
52	82681	17319	84307	15693	85913	14087	8
53	82708	17292	84334	15666	85940	14060	7
54	82735	17265	84361	15639	85967	14033	6
55	82762	17238	84388	15612	85993	14007	5
56	82790	17210	84415	15585	86020	13980	4
57	82817	17183	84442	15558	86046	13954	3
58	82844	17156	84469	15531	86073	13927	2
59	82871	17129	84496	15504	86100	13900	1
60	82899	17101	84523	15477	86126	13874	0
′	56° Cotan	56° Tan	55° Cotan	55° Tan	54° Cotan	54° Tan	′

′	36° Tan	36° Cotan	37° Tan	37° Cotan	38° Tan	38° Cotan	′
0	9.86126	10.13874	9.87711	10.12289	9.89281	10.10719	60
1	86153	13847	87738	12262	89307	10693	59
2	86179	13821	87764	12236	89333	10667	58
3	86206	13794	87790	12210	89359	10641	57
4	86232	13768	87817	12183	89385	10615	56
5	86259	13741	87843	12157	89411	10589	55
6	86285	13715	87869	12131	89437	10563	54
7	86312	13688	87895	12105	89463	10537	53
8	86338	13662	87922	12078	89489	10511	52
9	86365	13635	87948	12052	89515	10485	51
10	9.86392	10.13608	9.87974	10.12026	9.89541	10.10459	50
11	86418	13582	88000	12000	89567	10433	49
12	86445	13555	88027	11973	89593	10407	48
13	86471	13529	88053	11947	89619	10381	47
14	86498	13502	88079	11921	89645	10355	46
15	86524	13476	88105	11895	89671	10329	45
16	86551	13449	88131	11869	89697	10303	44
17	86577	13423	88158	11842	89723	10277	43
18	86603	13397	88184	11816	89749	10251	42
19	86630	13370	88210	11790	89775	10225	41
20	9.86656	10.13344	9.88236	10.11764	9.89801	10.10199	40
21	86683	13317	88262	11738	89827	10173	39
22	86709	13291	88289	11711	89853	10147	38
23	86736	13264	88315	11685	89879	10121	37
24	86762	13238	88341	11659	89905	10095	36
25	86789	13211	88367	11633	89931	10069	35
26	86815	13185	88393	11607	89957	10043	34
27	86842	13158	88420	11580	89983	10017	33
28	86868	13132	88446	11554	90009	09991	32
29	86894	13106	88472	11528	90035	09965	31
30	9.86921	10.13079	9.88498	10.11502	9.90061	10.09939	30
31	86947	13053	88524	11476	90086	09914	29
32	86974	13026	88550	11450	90112	09888	28
33	87000	13000	88577	11423	90138	09862	27
34	87027	12973	88603	11397	90164	09836	26
35	87053	12947	88629	11371	90190	09810	25
36	87079	12921	88655	11345	90216	09784	24
37	87106	12894	88681	11319	90242	09758	23
38	87132	12868	88707	11293	90268	09732	22
39	87158	12842	88733	11267	90294	09706	21
40	9.87185	10.12815	9.88759	10.11241	9.90320	10.09680	20
41	87211	12789	88786	11214	90346	09654	19
42	87238	12762	88812	11188	90371	09629	18
43	87264	12736	88838	11162	90397	09603	17
44	87290	12710	88864	11136	90423	09577	16
45	87317	12683	88890	11110	90449	09551	15
46	87343	12657	88916	11084	90475	09525	14
47	87369	12631	88942	11058	90501	09499	13
48	87396	12604	88968	11032	90527	09473	12
49	87422	12578	88994	11006	90553	09447	11
50	9.87448	10.12552	9.89020	10.10980	9.90578	10.09422	10
51	87475	12525	89046	10954	90604	09396	9
52	87501	12499	89073	10927	90630	09370	8
53	87527	12473	89099	10901	90656	09344	7
54	87554	12446	89125	10875	90682	09318	6
55	87580	12420	89151	10849	90708	09292	5
56	87606	12394	89177	10823	90734	09266	4
57	87633	12367	89203	10797	90759	09241	3
58	87659	12341	89229	10771	90785	09215	2
59	87685	12315	89255	10745	90811	09189	1
60	87711	12289	89281	10719	90837	09163	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	53°		52°		51°		

′	39° Tan	39° Cotan	40° Tan	40° Cotan	41° Tan	41° Cotan	′
0	9.90837	10.09163	9.92381	10.07619	9.93916	10.06084	60
1	90863	09137	92407	07593	93942	06058	59
2	90889	09111	92433	07567	93967	06033	58
3	90914	09086	92458	07542	93993	06007	57
4	90940	09060	92484	07516	94018	05982	56
5	90966	09034	92510	07490	94044	05956	55
6	90992	09008	92535	07465	94069	05931	54
7	91018	08982	92561	07439	94095	05905	53
8	91043	08957	92587	07413	94120	05880	52
9	91069	08931	92612	07388	94146	05854	51
10	9.91095	10.08905	9.92638	10.07362	9.94171	10.05829	50
11	91121	08879	92663	07337	94197	05803	49
12	91147	08853	92689	07311	94222	05778	48
13	91172	08828	92715	07285	94248	05752	47
14	91198	08802	92740	07260	94273	05727	46
15	91224	08776	92766	07234	94299	05701	45
16	91250	08750	92792	07208	94324	05676	44
17	91276	08724	92817	07183	94350	05650	43
18	91301	08699	92843	07157	94375	05625	42
19	91327	08673	92868	07132	94401	05599	41
20	9.91353	10.08647	9.92894	10.07106	9.94426	10.05574	40
21	91379	08621	92920	07080	94452	05548	39
22	91404	08596	92945	07055	94477	05523	38
23	91430	08570	92971	07029	94503	05497	37
24	91456	08544	92996	07004	94528	05472	36
25	91482	08518	93022	06978	94554	05446	35
26	91507	08493	93048	06952	94579	05421	34
27	91533	08467	93073	06927	94604	05396	33
28	91559	08441	93099	06901	94630	05370	32
29	91585	08415	93124	06876	94655	05345	31
30	9.91610	10.08390	9.93150	10.06850	9.94681	10.05319	30
31	91636	08364	93175	06825	94706	05294	29
32	91662	08338	93201	06799	94732	05268	28
33	91688	08312	93227	06773	94757	05243	27
34	91713	08287	93252	06748	94783	05217	26
35	91739	08261	93278	06722	94808	05192	25
36	91765	08235	93303	06697	94834	05166	24
37	91791	08209	93329	06671	94859	05141	23
38	91816	08184	93354	06646	94884	05116	22
39	91842	08158	93380	06620	94910	05090	21
40	9.91868	10.08132	9.93406	10.06594	9.94935	10.05065	20
41	91893	08107	93431	06569	94961	05039	19
42	91919	08081	93457	06543	94986	05014	18
43	91945	08055	93482	06518	95012	04988	17
44	91971	08029	93508	06492	95037	04963	16
45	91996	08004	93533	06467	95062	04938	15
46	92022	07978	93559	06441	95088	04912	14
47	92048	07952	93584	06416	95113	04887	13
48	92073	07927	93610	06390	95139	04861	12
49	92099	07901	93636	06364	95164	04836	11
50	9.92125	10.07875	9.93661	10.06339	9.95190	10.04810	10
51	92150	07850	93687	06313	95215	04785	9
52	92176	07824	93712	06288	95240	04760	8
53	92202	07798	93738	06262	95266	04734	7
54	92227	07773	93763	06237	95291	04709	6
55	92253	07747	93789	06211	95317	04683	5
56	92279	07721	93814	06186	95342	04658	4
57	92304	07696	93840	06160	95368	04632	3
58	92330	07670	93865	06135	95393	04607	2
59	92356	07644	93891	06109	95418	04582	1
60	92381	07619	93916	06084	95444	04556	0
′	50° Cotan	50° Tan	49° Cotan	49° Tan	48° Cotan	48° Tan	′

′	42°		43°		44°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	9.95444	10.04556	9.96966	10.03034	9.98484	10.01516	60
1	95469	04531	96991	03009	98509	01491	59
2	95495	04505	97016	02984	98534	01466	58
3	95520	04480	97042	02958	98560	01440	57
4	95545	04455	97067	02933	98585	01415	56
5	95571	04429	97092	02908	98610	01390	55
6	95596	04404	97118	02882	98635	01365	54
7	95622	04378	97143	02857	98661	01339	53
8	95647	04353	97168	02832	98686	01314	52
9	95672	04328	97193	02807	98711	01289	51
10	9.95698	10.04302	9.97219	10.02781	9.98737	10.01263	50
11	95723	04277	97244	02756	98762	01238	49
12	95748	04252	97269	02731	98787	01213	48
13	95774	04226	97295	02705	98812	01188	47
14	95799	04201	97320	02680	98838	01162	46
15	95825	04175	97345	02655	98863	01137	45
16	95850	04150	97371	02629	98888	01112	44
17	95875	04125	97396	02604	98913	01087	43
18	95901	04099	97421	02579	98939	01061	42
19	95926	04074	97447	02553	98964	01086	41
20	9.95952	10.04048	9.97472	10.02528	9.98989	10.01011	40
21	95977	04023	97497	02503	99015	00985	39
22	96002	03998	97523	02477	99040	00960	38
23	96028	03972	97548	02452	99065	00935	37
24	96053	03947	97573	02427	99090	00910	36
25	96078	03922	97598	02402	99116	00884	35
26	96104	03896	97624	02376	99141	00859	34
27	96129	03871	97649	02351	99166	00834	33
28	96155	03845	97674	02326	99191	00809	32
29	96180	03820	97700	02300	99217	00783	31
30	9.96205	10.03795	9.97725	10.02275	9.99242	10.00758	30
31	96231	03769	97750	02250	99267	00733	29
32	96256	03744	97776	02224	99293	00707	28
33	96281	03719	97801	02199	99318	00682	27
34	96307	03693	97826	02174	99343	00657	26
35	96332	03668	97851	02149	99368	00632	25
36	96357	03643	97877	02123	99394	00606	24
37	96383	03617	97902	02098	99419	00581	23
38	96408	03592	97927	02073	99444	00556	22
39	96433	03567	97953	02047	99469	00531	21
40	9.96459	10.03541	9.97978	10.02022	9.99495	10.00505	20
41	96484	03516	98003	01997	99520	00480	19
42	96510	03490	98029	01971	99545	00455	18
43	96535	03465	98054	01946	99570	00430	17
44	96560	03440	98079	01921	99596	00404	16
45	96586	03414	98104	01896	99621	00379	15
46	96611	03389	98130	01870	99646	00354	14
47	96636	03364	98155	01845	99672	00328	13
48	96662	03338	98180	01820	99697	00303	12
49	96687	03313	98206	01794	99722	00278	11
50	9.96712	10.03288	9.98231	10.01769	9.99747	10.00253	10
51	96738	03262	98256	01744	99773	00227	9
52	96763	03237	98281	01719	99798	00202	8
53	96788	03212	98307	01693	99823	00177	7
54	96814	03186	98332	01668	99848	00152	6
55	96839	03161	98357	01643	99874	00126	5
56	96864	03136	98383	01617	99899	00101	4
57	96890	03110	98408	01592	99924	00076	3
58	96915	03085	98433	01567	99949	00051	2
59	96940	03060	98458	01542	99975	00025	1
60	96966	03034	98484	01516	10.00000	00000	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	47°		46°		45°		

TABLE XIII NATURAL SINES AND COSINES

′	0°		1°		2°		3°		4°		′
	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	
0	.00000	One.	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.00029	One.	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.00058	One.	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.00087	One.	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.00116	One.	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.00145	One.	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.00175	One.	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.00204	One.	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.00233	One.	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.00262	One.	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.00291	One.	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.00320	.99999	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.00349	.99999	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.00378	.99999	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.00407	.99999	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.00436	.99999	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.00465	.99999	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.00495	.99999	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.00524	.99999	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.00553	.99998	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.00611	.99998	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.00640	.99998	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.00669	.99998	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.00698	.99998	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.00727	.99997	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.00756	.99997	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.00785	.99997	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.00814	.99997	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.00844	.99996	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.00902	.99996	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.00931	.99996	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.00960	.99995	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.00989	.99995	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.01018	.99995	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.01047	.99995	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.01076	.99994	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.01105	.99994	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.01134	.99994	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.01193	.99993	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.01222	.99993	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.01251	.99992	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.01280	.99992	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.01309	.99991	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.01338	.99991	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.01367	.99991	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.01396	.99990	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.01425	.99990	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.01483	.99989	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.01513	.99989	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.01542	.99988	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.01571	.99988	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.01600	.99987	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.01629	.99987	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.01658	.99986	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.01687	.99986	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.01716	.99985	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	′
	89°		88°		87°		86°		85°		

′	5°		6°		7°		8°		9°		′
	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	.15672	.98764	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	.15701	.98760	58
3	.08803	99612	.10540	.99443	.12274	.99244	.14004	.99015	.15730	.98755	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	.15758	.98751	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	.15787	.98746	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	.15816	.98741	54
7	.08918	.99602	.10655	.99431	.12389	.99230	.14119	.98998	.15845	.98737	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	.15873	.98732	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	.15902	.98728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	.15959	.98718	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	.15988	.98714	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	.16017	.98709	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	.16046	.98704	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	.16074	.98700	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	.16103	.98695	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	.16132	.98690	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	.16160	.98686	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	.16189	.98681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	.98676	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	.16246	.98671	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	.16275	.98667	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	.16304	.98662	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	.16333	.98657	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	.16361	.98652	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	.16390	.98648	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	.16419	.98643	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	.16447	.98638	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	.16476	.98633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	.16533	.98624	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	.16562	.98619	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	.16591	.98614	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	.16620	.98609	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	.16648	.98604	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	.16677	.98600	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	.16706	.98595	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	.16734	.98590	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	.16763	.98585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	.16820	.98575	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	.16849	.98570	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	.16878	.98565	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	.16906	.98561	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	.16935	.98556	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	.16964	.98551	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	.16992	.98546	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	.17021	.98541	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	.17050	.98536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	.17107	.98526	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	.17136	.98521	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	.17164	.98516	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	.17193	.98511	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	.17222	.98506	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	.17250	.98501	4
57	.10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	.17279	.98496	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	.17308	.98491	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	.17336	.98486	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	.17365	.98481	0
′	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	′
	84°		83°		82°		81°		80°		

′	10° Sin	10° Cosin	11° Sin	11° Cosin	12° Sin	12° Cosin	13° Sin	13° Cosin	14° Sin	14° Cosin	′
0	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	**60**
1	.17393	.98476	.19109	.98157	.20820	.97809	.22523	.97430	.24220	.97023	**59**
2	.17422	.98471	.19138	.98152	.20848	.97803	.22552	.97424	.24249	.97015	**58**
3	.17451	.98466	.19167	.98146	.20877	.97797	.22580	.97417	.24277	.97008	**57**
4	.17479	.98461	.19195	.98140	.20905	.97791	.22608	.97411	.24305	.97001	**56**
5	.17508	.98455	.19224	.98135	.20933	.97784	.22637	.97404	.24333	.96994	**55**
6	.17537	.98450	.19252	.98129	.20962	.97778	.22665	.97398	.24362	.96987	**54**
7	.17565	.98445	.19281	.98124	.20990	.97772	.22693	.97391	.24390	.96980	**53**
8	.17594	.98440	.19309	.98118	.21019	.97766	.22722	.97384	.24418	.96973	**52**
9	.17623	.98435	.19338	.98112	.21047	.97760	.22750	.97378	.24446	.96966	**51**
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97371	.24474	.96959	**50**
11	.17680	.98425	.19395	.98101	.21104	.97748	.22807	.97365	.24503	.96952	**49**
12	.17708	.98420	.19423	.98096	.21132	.97742	.22835	.97358	.24531	.96945	**48**
13	.17737	.98414	.19452	.98090	.21161	.97735	.22863	.97351	.24559	.96937	**47**
14	.17766	.98409	.19481	.98084	.21189	.97729	.22892	.97345	.24587	.96930	**46**
15	.17794	.98404	.19509	.98079	.21218	.97723	.22920	.97338	.24615	.96923	**45**
16	.17823	.98399	.19538	.98073	.21246	.97717	.22948	.97331	.24644	.96916	**44**
17	.17852	.98394	.19566	.98067	.21275	.97711	.22977	.97325	.24672	.96909	**43**
18	.17880	.98389	.19595	.98061	.21303	.97705	.23005	.97318	.24700	.96902	**42**
19	.17909	.98383	.19623	.98056	.21331	.97698	.23033	.97311	.24728	.96894	**41**
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	**40**
21	.17966	.98373	.19680	.98044	.21388	.97686	.23090	.97298	.24784	.96880	**39**
22	.17995	.98368	.19709	.98039	.21417	.97680	.23118	.97291	.24813	.96873	**38**
23	.18023	.98362	.19737	.98033	.21445	.97673	.23146	.97284	.24841	.96866	**37**
24	.18052	.98357	.19766	.98027	.21474	.97667	.23175	.97278	.24869	.96858	**36**
25	.18081	.98352	.19794	.98021	.21502	.97661	.23203	.97271	.24897	.96851	**35**
26	.18109	.98347	.19823	.98016	.21530	.97655	.23231	.97264	.24925	.96844	**34**
27	.18138	.98341	.19851	.98010	.21559	.97648	.23260	.97257	.24954	.96837	**33**
28	.18166	.98336	.19880	.98004	.21587	.97642	.23288	.97251	.24982	.96829	**32**
29	.18195	.98331	.19908	.97998	.21616	.97636	.23316	.97244	.25010	.96822	**31**
30	.18224	.98325	.19937	.97992	.21644	.97630	.23345	.97237	.25038	.96815	**30**
31	.18252	.98320	.19965	.97987	.21672	.97623	.23373	.97230	.25066	.96807	**29**
32	.18281	.98315	.19994	.97981	.21701	.97617	.23401	.97223	.25094	.96800	**28**
33	.18309	.98310	.20022	.97975	.21729	.97611	.23429	.97217	.25122	.96793	**27**
34	.18338	.98304	.20051	.97969	.21758	.97604	.23458	.97210	.25151	.96786	**26**
35	.18367	.98299	.20079	.97963	.21786	.97598	.23486	.97203	.25179	.96778	**25**
36	.18395	.98294	.20108	.97958	.21814	.97592	.23514	.97196	.25207	.96771	**24**
37	.18424	.98288	.20136	.97952	.21843	.97585	.23542	.97189	.25235	.96764	**23**
38	.18452	.98283	.20165	.97946	.21871	.97579	.23571	.97182	.25263	.96756	**22**
39	.18481	.98277	.20193	.97940	.21899	.97573	.23599	.97176	.25291	.96749	**21**
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	**20**
41	.18538	.98267	.20250	.97928	.21956	.97560	.23656	.97162	.25348	.96734	**19**
42	.18567	.98261	.20279	.97922	.21985	.97553	.23684	.97155	.25376	.96727	**18**
43	.18595	.98256	.20307	.97916	.22013	.97547	.23712	.97148	.25404	.96719	**17**
44	.18624	.98250	.20336	.97910	.22041	.97541	.23740	.97141	.25432	.96712	**16**
45	.18652	.98245	.20364	.97905	.22070	.97534	.23769	.97134	.25460	.96705	**15**
46	.18681	.98240	.20393	.97899	.22098	.97528	.23797	.97127	.25488	.96697	**14**
47	.18710	.98234	.20421	.97893	.22126	.97521	.23825	.97120	.25516	.96690	**13**
48	.18738	.98229	.20450	.97887	.22155	.97515	.23853	.97113	.25545	.96682	**12**
49	.18767	.98223	.20478	.97881	.22183	.97508	.23882	.97106	.25573	.96675	**11**
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	**10**
51	.18824	.98212	.20535	.97869	.22240	.97496	.23938	.97093	.25629	.96660	**9**
52	.18852	.98207	.20563	.97863	.22268	.97489	.23966	.97086	.25657	.96653	**8**
53	.18881	.98201	.20592	.97857	.22297	.97483	.23995	.97079	.25685	.96645	**7**
54	.18910	.98196	.20620	.97851	.22325	.97476	.24023	.97072	.25713	.96638	**6**
55	.18938	.98190	.20649	.97845	.22353	.97470	.24051	.97065	.25741	.96630	**5**
56	.18967	.98185	.20677	.97839	.22382	.97463	.24079	.97058	.25769	.96623	**4**
57	.18995	.98179	.20706	.97833	.22410	.97457	.24108	.97051	.25798	.96615	**3**
58	.19024	.98174	.20734	.97827	.22438	.97450	.24136	.97044	.25826	.96608	**2**
59	.19052	.98168	.20763	.97821	.22467	.97444	.24164	.97037	.25854	.96600	**1**
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	**0**
′	Cosin 79°	Sin 79°	Cosin 78°	Sin 78°	Cosin 77°	Sin 77°	Cosin 76°	Sin 76°	Cosin 75°	Sin 75°	′

′	15° Sin	15° Cosin	16° Sin	16° Cosin	17° Sin	17° Cosin	18° Sin	18° Cosin	19° Sin	19° Cosin	′
0	.25882	.96593	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	60
1	.25910	.96585	.27592	.96118	.29265	.95622	.30929	.95097	.32584	.94542	59
2	.25938	.96578	.27620	.96110	.29293	.95613	.30957	.95088	.32612	.94533	58
3	.25966	.96570	.27648	.96102	.29321	.95605	.30985	.95079	.32639	.94523	57
4	.25994	.96562	.27676	.96094	.29348	.95596	.31012	.95070	.32667	.94514	56
5	.26022	.96555	.27704	.96086	.29376	.95588	.31040	.95061	.32694	.94504	55
6	.26050	.96547	.27731	.96078	.29404	.95579	.31068	.95052	.32722	.94495	54
7	.26079	.96540	.27759	.96070	.29432	.95571	.31095	.95043	.32749	.94485	53
8	.26107	.96532	.27787	.96062	.29460	.95562	.31123	.95033	.32777	.94476	52
9	.26135	.96524	.27815	.96054	.29487	.95554	.31151	.95024	.32804	.94466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	50
11	.26191	.96509	.27871	.96037	.29543	.95536	.31206	.95006	.32859	.94447	49
12	.26219	.96502	.27899	.96029	.29571	.95528	.31233	.94997	.32887	.94438	48
13	.26247	.96494	.27927	.96021	.29599	.95519	.31261	.94988	.32914	.94428	47
14	.26275	.96486	.27955	.96013	.29626	.95511	.31289	.94979	.32942	.94418	46
15	.26303	.96479	.27983	.96005	.29654	.95502	.31316	.94970	.32969	.94409	45
16	.26331	.96471	.28011	.95997	.29682	.95493	.31344	.94961	.32997	.94399	44
17	.26359	.96463	.28039	.95989	.29710	.95485	.31372	.94952	.33024	.94390	43
18	.26387	.96456	.28067	.95981	.29737	.95476	.31399	.94943	.33051	.94380	42
19	.26415	.96448	.28095	.95972	.29765	.95467	.31427	.94933	.33079	.94370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	40
21	.26471	.96433	.28150	.95956	.29821	.95450	.31482	.94915	.33134	.94351	39
22	.26500	.96425	.28178	.95948	.29849	.95441	.31510	.94906	.33161	.94342	38
23	.26528	.96417	.28206	.95940	.29876	.95433	.31537	.94897	.33189	.94332	37
24	.26556	.96410	.28234	.95931	.29904	.95424	.31565	.94888	.33216	.94322	36
25	.26584	.96402	.28262	.95923	.29932	.95415	.31593	.94878	.33244	.94313	35
26	.26612	.96394	.28290	.95915	.29960	.95407	.31620	.94869	.33271	.94303	34
27	.26640	.96386	.28318	.95907	.29987	.95398	.31648	.94860	.33298	.94293	33
28	.26668	.96379	.28346	.95898	.30015	.95389	.31675	.94851	.33326	.94284	32
29	.26696	.96371	.28374	.95890	.30043	.95380	.31703	.94842	.33353	.94274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	30
31	.26752	.96355	.28429	.95874	.30098	.95363	.31758	.94823	.33408	.94254	29
32	.26780	.96347	.28457	.95865	.30126	.95354	.31786	.94814	.33436	.94245	28
33	.26808	.96340	.28485	.95857	.30154	.95345	.31813	.94805	.33463	.94235	27
34	.26836	.96332	.28513	.95849	.30182	.95337	.31841	.94795	.33490	.94225	26
35	.26864	.96324	.28541	.95841	.30209	.95328	.31868	.94786	.33518	.94215	25
36	.26892	.96316	.28569	.95832	.30237	.95319	.31896	.94777	.33545	.94206	24
37	.26920	.96308	.28597	.95824	.30265	.95310	.31923	.94768	.33573	.94196	23
38	.26948	.96301	.28625	.95816	.30292	.95301	.31951	.94758	.33600	.94186	22
39	.26976	.96293	.28652	.95807	.30320	.95293	.31979	.94749	.33627	.94176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	20
41	.27032	.96277	.28708	.95791	.30376	.95275	.32034	.94730	.33682	.94157	19
42	.27060	.96269	.28736	.95782	.30403	.95266	.32061	.94721	.33710	.94147	18
43	.27088	.96261	.28764	.95774	.30431	.95257	.32089	.94712	.33737	.94137	17
44	.27116	.96253	.28792	.95766	.30459	.95248	.32116	.94702	.33764	.94127	16
45	.27144	.96246	.28820	.95757	.30486	.95240	.32144	.94693	.33792	.94118	15
46	.27172	.96238	.28847	.95749	.30514	.95231	.32171	.94684	.33819	.94108	14
47	.27200	.96230	.28875	.95740	.30542	.95222	.32199	.94674	.33846	.94098	13
48	.27228	.96222	.28903	.95732	.30570	.95213	.32227	.94665	.33874	.94088	12
49	.27256	.96214	.28931	.95724	.30597	.95204	.32254	.94656	.33901	.94078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	10
51	.27312	.96198	.28987	.95707	.30653	.95186	.32309	.94637	.33956	.94058	9
52	.27340	.96190	.29015	.95698	.30680	.95177	.32337	.94627	.33983	.94049	8
53	.27368	.96182	.29042	.95690	.30708	.95168	.32364	.94618	.34011	.94039	7
54	.27396	.96174	.29070	.95681	.30736	.95159	.32392	.94609	.34038	.94029	6
55	.27424	.96166	.29098	.95673	.30763	.95150	.32419	.94599	.34065	.94019	5
56	.27452	.96158	.29126	.95664	.30791	.95142	.32447	.94590	.34093	.94009	4
57	.27480	.96150	.29154	.95656	.30819	.95133	.32474	.94580	.34120	.93999	3
58	.27508	.96142	.29182	.95647	.30846	.95124	.32502	.94571	.34147	.93989	2
59	.27536	.96134	.29209	.95639	.30874	.95115	.32529	.94561	.34175	.93979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	0
′	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	′
	74°		73°		72°		71°		70°		

′	20°		21°		22°		23°		24°		′
	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	
0	.34202	.93969	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	.34229	.93959	.35864	.93348	.37488	.92707	.39100	.92039	.40700	.91343	59
2	.34257	.93949	.35891	.93337	.37515	.92697	.39127	.92028	.40727	.91331	58
3	.34284	.93939	.35918	.93327	.37542	.92686	.39153	.92016	.40753	.91319	57
4	.34311	.93929	.35945	.93316	.37569	.92675	.39180	.92005	.40780	.91307	56
5	.34339	.93919	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	.34366	.93909	.36000	.93295	.37622	.92653	.39234	.91982	.40833	.91283	54
7	.34393	.93899	.36027	.93285	.37649	.92642	.39260	.91971	.40860	.91272	53
8	.34421	.93889	.36054	.93274	.37676	.92631	.39287	.91959	.40886	.91260	52
9	.34448	.93879	.36081	.93264	.37703	.92620	.39314	.91948	.40913	.91248	51
10	.34475	.93869	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	.34503	.93859	.36135	.93243	.37757	.92598	.39367	.91925	.40966	.91224	49
12	.34530	.93849	.36162	.93232	.37784	.92587	.39394	.91914	.40992	.91212	48
13	.34557	.93839	.36190	.93222	.37811	.92576	.39421	.91902	.41019	.91200	47
14	.34584	.93829	.36217	.93211	.37838	.92565	.39448	.91891	.41045	.91188	46
15	.34612	.93819	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	.34639	.93809	.36271	.93190	.37892	.92543	.39501	.91868	.41098	.91164	44
17	.34666	.93799	.36298	.93180	.37919	.92532	.39528	.91856	.41125	.91152	43
18	.34694	.93789	.36325	.93169	.37946	.92521	.39555	.91845	.41151	.91140	42
19	.34721	.93779	.36352	.93159	.37973	.92510	.39581	.91833	.41178	.91128	41
20	.34748	.93769	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	.34775	.93750	.36406	.93137	.38026	.92488	.39635	.91810	.41231	.91104	39
22	.34803	.93748	.36434	.93127	.38053	.92477	.39661	.91799	.41257	.91092	38
23	.34830	.93738	.36461	.93116	.38080	.92466	.39688	.91787	.41284	.91080	37
24	.34857	.93728	.36488	.93106	.38107	.92455	.39715	.91775	.41310	.91068	36
25	.34884	.93718	.36515	.93095	.38134	.92444	.39741	.91764	.41337	.91056	35
26	.34912	.93708	.36542	.93084	.38161	.92432	.39768	.91752	.41363	.91044	34
27	.34939	.93698	.36569	.93074	.38188	.92421	.39795	.91741	.41390	.91032	33
28	.34966	.93688	.36596	.93063	.38215	.92410	.39822	.91729	.41416	.91020	32
29	.34993	.93677	.36623	.93052	.38241	.92399	.39848	.91718	.41443	.91008	31
30	.35021	.93667	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	.35048	.93657	.36677	.93031	.38295	.92377	.39902	.91694	.41496	.90984	29
32	.35075	.93647	.36704	.93020	.38322	.92366	.39928	.91683	.41522	.90972	28
33	.35102	.93637	.36731	.93010	.38349	.92355	.39955	.91671	.41549	.90960	27
34	.35130	.93626	.36758	.92999	.38376	.92343	.39982	.91660	.41575	.90948	26
35	.35157	.93616	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	.35184	.93606	.36812	.92978	.38430	.92321	.40035	.91636	.41628	.90924	24
37	.35211	.93596	.36839	.92967	.38456	.92310	.40062	.91625	.41655	.90911	23
38	.35239	.93585	.36867	.92956	.38483	.92299	.40088	.91613	.41681	.90899	22
39	.35266	.93575	.36894	.92945	.38510	.92287	.40115	.91601	.41707	.90887	21
40	.35293	.93565	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	.35320	.93555	.36948	.92924	.38564	.92265	.40168	.91578	.41760	.90863	19
42	.35347	.93544	.36975	.92913	.38591	.92254	.40195	.91566	.41787	.90851	18
43	.35375	.93534	.37002	.92902	.38617	.92243	.40221	.91555	.41813	.90839	17
44	.35402	.93524	.37029	.92892	.38644	.92231	.40248	.91543	.41840	.90826	16
45	.35429	.93514	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	.35456	.93503	.37083	.92870	.38698	.92209	.40301	.91519	.41892	.90802	14
47	.35484	.93493	.37110	.92859	.38725	.92198	.40328	.91508	.41919	.90790	13
48	.35511	.93483	.37137	.92849	.38752	.92186	.40355	.91496	.41945	.90778	12
49	.35538	.93472	.37164	.92838	.38778	.92175	.40381	.91484	.41972	.90766	11
50	.35565	.93462	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	.35592	.93452	.37218	.92816	.38832	.92152	.40434	.91461	.42024	.90741	9
52	.35619	.93441	.37245	.92805	.38859	.92141	.40461	.91449	.42051	.90729	8
53	.35647	.93431	.37272	.92794	.38886	.92130	.40488	.91437	.42077	.90717	7
54	.35674	.93420	.37299	.92784	.38912	.92119	.40514	.91425	.42104	.90704	6
55	.35701	.93410	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	.35728	.93400	.37353	.92762	.38966	.92096	.40567	.91402	.42156	.90680	4
57	.35755	.93389	.37380	.92751	.38993	.92085	.40594	.91390	.42183	.90668	3
58	.35782	.93379	.37407	.92740	.39020	.92073	.40621	.91378	.42209	.90655	2
59	.35810	.93368	.37434	.92729	.39046	.92062	.40647	.91366	.42235	.90643	1
60	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
′	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	′
	69°		68°		67°		66°		65°		

′	25° Sin	25° Cosin	26° Sin	26° Cosin	27° Sin	27° Cosin	28° Sin	28° Cosin	29° Sin	29° Cosin	′
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	.48506	.87448	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	.48532	.87434	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	.48557	.87420	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	.48583	.87406	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	.48608	.87391	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	.48634	.87377	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	.48659	.87363	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	.48684	.87349	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	.48710	.87335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	50
11	.42552	.90495	.44124	.89739	.45684	.88955	.47229	.88144	.48761	.87306	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	.48786	.87292	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	.48811	.87278	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	.48837	.87264	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	.48862	.87250	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	.48888	.87235	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	.48913	.87221	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	.48938	.87207	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	.48964	.87193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	.49014	.87164	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	.49040	.87150	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	.49065	.87136	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	.49090	.87121	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	.49116	.87107	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	.49141	.87093	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	.49166	.87079	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	.49192	.87064	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	.49217	.87050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	.49268	.87021	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	.49293	.87007	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	.49318	.86993	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	.49344	.86978	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	.49369	.86964	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	.49394	.86949	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	.49419	.86935	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	.49445	.86921	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	.49470	.86906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	.49521	.86878	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	.49546	.86863	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	.49571	.86849	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	.49596	.86834	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	.49622	.86820	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	.49647	.86805	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	.49672	.86791	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	.49697	.86777	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	.49723	.86762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	.49773	.86733	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	.49798	.86719	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	.49824	.86704	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	.49849	.86690	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	.49874	.86675	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	.49899	.86661	4
57	.43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	.49924	.86646	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	.49950	.86632	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	.49975	.86617	1
60	.43837	.89879	.45399	.89101	46947	.88295	.48481	.87462	.50000	.86603	0
′	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	′
	64°		63°		62°		61°		60°		

TABLE XIII NATURAL SINES AND COSINES *(Continued)*

′	30°		31°		32°		33°		34°		′
	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	
0	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	60
1	.50025	.86588	.51529	.85702	.53017	.84789	.54488	.83851	.55943	.82887	59
2	.50050	.86573	.51554	.85687	.53041	.84774	.54513	.83835	.55968	.82871	58
3	.50076	.86559	.51579	.85672	.53066	.84759	.54537	.83819	.55992	.82855	57
4	.50101	.86544	.51604	.85657	.53091	.84743	.54561	.83804	.56016	.82839	56
5	.50126	.86530	.51628	.85642	.53115	.84728	.54586	.83788	.56040	.82822	55
6	.50151	.86515	.51653	.85627	.53140	.84712	.54610	.83772	.56064	.82806	54
7	.50176	.86501	.51678	.85612	.53164	.84697	.54635	.83756	.56088	.82790	53
8	.50201	.86486	.51703	.85597	.53189	.84681	.54659	.83740	.56112	.82773	52
9	.50227	.86471	.51728	.85582	.53214	.84666	.54683	.83724	.56136	.82757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	50
11	.50277	.86442	.51778	.85551	.53263	.84635	.54732	.83692	.56184	.82724	49
12	.50302	.86427	.51803	.85536	.53288	.84619	.54756	.83676	.56208	.82708	48
13	.50327	.86413	.51828	.85521	.53312	.84604	.54781	.83660	.56232	.82692	47
14	.50352	.86398	.51852	.85506	.53337	.84588	.54805	.83645	.56256	.82675	46
15	.50377	.86384	.51877	.85491	.53361	.84573	.54829	.83629	.56280	.82659	45
16	.50403	.86369	.51902	.85476	.53386	.84557	.54854	.83613	.56305	.82643	44
17	.50428	.86354	.51927	.85461	.53411	.84542	.54878	.83597	.56329	.82626	43
18	.50453	.86340	.51952	.85446	.53435	.84526	.54902	.83581	.56353	.82610	42
19	.50478	.86325	.51977	.85431	.53460	.84511	.54927	.83565	.56377	.82593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	40
21	.50528	.86295	.52026	.85401	.53509	.84480	.54975	.83533	.56425	.82561	39
22	.50553	.86281	.52051	.85385	.53534	.84464	.54999	.83517	.56449	.82544	38
23	.50578	.86266	.52076	.85370	.53558	.84448	.55024	.83501	.56473	.82528	37
24	.50603	.86251	.52101	.85355	.53583	.84433	.55048	.83485	.56497	.82511	36
25	.50628	.86237	.52126	.85340	.53607	.84417	.55072	.83469	.56521	.82495	35
26	.50654	.86222	.52151	.85325	.53632	.84402	.55097	.83453	.56545	.82478	34
27	.50679	.86207	.52175	.85310	.53656	.84386	.55121	.83437	.56569	.82462	33
28	.50704	.86192	.52200	.85294	.53681	.84370	.55145	.83421	.56593	.82446	32
29	.50729	.86178	.52225	.85279	.53705	.84355	.55169	.83405	.56617	.82429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	30
31	.50779	.86148	.52275	.85249	.53754	.84324	.55218	.83373	.56665	.82396	29
32	.50804	.86133	.52299	.85234	.53779	.84308	.55242	.83356	.56689	.82380	28
33	.50829	.86119	.52324	.85218	.53804	.84292	.55266	.83340	.56713	.82363	27
34	.50854	.86104	.52349	.85203	.53828	.84277	.55291	.83324	.56736	.82347	26
35	.50879	.86089	.52374	.85188	.53853	.84261	.55315	.83308	.56760	.82330	25
36	.50904	.86074	.52399	.85173	.53877	.84245	.55339	.83292	.56784	.82314	24
37	.50929	.86059	.52423	.85157	.53902	.84230	.55363	.83276	.56808	.82297	23
38	.50954	.86045	.52448	.85142	.53926	.84214	.55388	.83260	.56832	.82281	22
39	.50979	.86030	.52473	.85127	.53951	.84198	.55412	.83244	.56856	.82264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	20
41	.51029	.86000	.52522	.85096	.54000	.84167	.55460	.83212	.56904	.82231	19
42	.51054	.85985	.52547	.85081	.54024	.84151	.55484	.83195	.56928	.82214	18
43	.51079	.85970	.52572	.85066	.54049	.84135	.55509	.83179	.56952	.82198	17
44	.51104	.85956	.52597	.85051	.54073	.84120	.55533	.83163	.56976	.82181	16
45	.51129	.85941	.52621	.85035	.54097	.84104	.55557	.83147	.57000	.82165	15
46	.51154	.85926	.52646	.85020	.54122	.84088	.55581	.83131	.57024	.82148	14
47	.51179	.85911	.52671	.85005	.54146	.84072	.55605	.83115	.57047	.82132	13
48	.51204	.85896	.52696	.84989	.54171	.84057	.55630	.83098	.57071	.82115	12
49	.51229	.85881	.52720	.84974	.54195	.84041	.55654	.83082	.57095	.82098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	10
51	.51279	.85851	.52770	.84943	.54244	.84009	.55702	.83050	.57143	.82065	9
52	.51304	.85836	.52794	.84928	.54269	.83994	.55726	.83034	.57167	.82048	8
53	.51329	.85821	.52819	.84913	.54293	.83978	.55750	.83017	.57191	.82032	7
54	.51354	.85806	.52844	.84897	.54317	.83962	.55775	.83001	.57215	.82015	6
55	.51379	.85792	.52869	.84882	.54342	.83946	.55799	.82985	.57238	.81999	5
56	.51404	.85777	.52893	.84866	.54366	.83930	.55823	.82969	.57262	.81982	4
57	.51429	.85762	.52918	.84851	.54391	.83915	.55847	.82953	.57286	.81965	3
58	.51454	.85747	.52943	.84836	.54415	.83899	.55871	.82936	.57310	.81949	2
59	.51479	.85732	.52967	.84820	.54440	.83883	.55895	.82920	.57334	.81932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	0
′	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	′
	59°		58°		57°		56°		55°		

′	35°		36°		37°		38°		39°		′
	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	
0	.57358	.81915	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	60
1	.57381	.81899	.58802	.80885	.60205	.79846	.61589	.78783	.62955	.77696	59
2	.57405	.81882	.58826	.80867	.60228	.79829	.61612	.78765	.62977	.77678	58
3	.57429	.81865	.58849	.80850	.60251	.79811	.61635	.78747	.63000	.77660	57
4	.57453	.81848	.58873	.80833	.60274	.79793	.61658	.78729	.63022	.77641	56
5	.57477	.81832	.58896	.80816	.60298	.79776	.61681	.78711	.63045	.77623	55
6	.57501	.81815	.58920	.80799	.60321	.79758	.61704	.78694	.63068	.77605	54
7	.57524	.81798	.58943	.80782	.60344	.79741	.61726	.78676	.63090	.77586	53
8	.57548	.81782	.58967	.80765	.60367	.79723	.61749	.78658	.63113	.77568	52
9	.57572	.81765	.58990	.80748	.60390	.79706	.61772	.78640	.63135	.77550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.63158	.77531	50
11	.57619	.81731	.59037	.80713	.60437	.79671	.61818	.78604	.63180	.77513	49
12	.57643	.81714	.59061	.80696	.60460	.79653	.61841	.78586	.63203	.77494	48
13	.57667	.81698	.59084	.80679	.60483	.79635	.61864	.78568	.63225	.77476	47
14	.57691	.81681	.59108	.80662	.60506	.79618	.61887	.78550	.63248	.77458	46
15	.57715	.81664	.59131	.80644	.60529	.79600	.61909	.78532	.63271	.77439	45
16	.57738	.81647	.59154	.80627	.60553	.79583	.61932	.78514	.63293	.77421	44
17	.57762	.81631	.59178	.80610	.60576	.79565	.61955	.78496	.63316	.77402	43
18	.57786	.81614	.59201	.80593	.60599	.79547	.61978	.78478	.63338	.77384	42
19	.57810	.81597	.59225	.80576	.60622	.79530	.62001	.78460	.63361	.77366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	40
21	.57857	.81563	.59272	.80541	.60668	.79494	.62046	.78424	.63406	.77329	39
22	.57881	.81546	.59295	.80524	.60691	.79477	.62069	.78405	.63428	.77310	38
23	.57904	.81530	.59318	.80507	.60714	.79459	.62092	.78387	.63451	.77292	37
24	.57928	.81513	.59342	.80489	.60738	.79441	.62115	.78369	.63473	.77273	36
25	.57952	.81496	.59365	.80472	.60761	.79424	.62138	.78351	.63496	.77255	35
26	.57976	.81479	.59389	.80455	.60784	.79406	.62160	.78333	.63518	.77236	34
27	.57999	.81462	.59412	.80438	.60807	.79388	.62183	.78315	.63540	.77218	33
28	.58023	.81445	.59436	.80420	.60830	.79371	.62206	.78297	.63563	.77199	32
29	.58047	.81428	.59459	.80403	.60853	.79353	.62229	.78279	.63585	.77181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	30
31	.58094	.81395	.59506	.80368	.60899	.79318	.62274	.78243	.63630	.77144	29
32	.58118	.81378	.59529	.80351	.60922	.79300	.62297	.78225	.63653	.77125	28
33	.58141	.81361	.59552	.80334	.60945	.79282	.62320	.78206	.63675	.77107	27
34	.58165	.81344	.59576	.80316	.60968	.79264	.62342	.78188	.63698	.77088	26
35	.58189	.81327	.59599	.80299	.60991	.79247	.62365	.78170	.63720	.77070	25
36	.58212	.81310	.59622	.80282	.61015	.79229	.62388	.78152	.63742	.77051	24
37	.58236	.81293	.59646	.80264	.61038	.79211	.62411	.78134	.63765	.77033	23
38	.58260	.81276	.59669	.80247	.61061	.79193	.62433	.78116	.63787	.77014	22
39	.58283	.81259	.59693	.80230	.61084	.79176	.62456	.78098	.63810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	20
41	.58330	.81225	.59739	.80195	.61130	.79140	.62502	.78061	.63854	.76959	19
42	.58354	.81208	.59763	.80178	.61153	.79122	.62524	.78043	.63877	.76940	18
43	.58378	.81191	.59786	.80160	.61176	.79105	.62547	.78025	.63899	.76921	17
44	.58401	.81174	.59809	.80143	.61199	.79087	.62570	.78007	.63922	.76903	16
45	.58425	.81157	.59832	.80125	.61222	.79069	.62592	.77988	.63944	.76884	15
46	.58449	.81140	.59856	.80108	.61245	.79051	.62615	.77970	.63966	.76866	14
47	.58472	.81123	.59879	.80091	.61268	.79033	.62638	.77952	.63989	.76847	13
48	.58496	.81106	.59902	.80073	.61291	.79016	.62660	.77934	.64011	.76828	12
49	.58519	.81089	.59926	.80056	.61314	.78998	.62683	.77916	.64033	.76810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	10
51	.58567	.81055	.59972	.80021	.61360	.78962	.62728	.77879	.64078	.76772	9
52	.58590	.81038	.59995	.80003	.61383	.78944	.62751	.77861	.64100	.76754	8
53	.58614	.81021	.60019	.79986	.61406	.78926	.62774	.77843	.64123	.76735	7
54	.58637	.81004	.60042	.79968	.61429	.78908	.62796	.77824	.64145	.76717	6
55	.58661	.80987	.60065	.79951	.61451	.78891	.62819	.77806	.64167	.76698	5
56	.58684	.80970	.60089	.79934	.61474	.78873	.62842	.77788	.64190	.76679	4
57	.58708	.80953	.60112	.79916	.61497	.78855	.62864	.77769	.64212	.76661	3
58	.58731	.80936	.60135	.79899	.61520	.78837	.62887	.77751	.64234	.76642	2
59	.58755	.80919	.60158	.79881	.61543	.78819	.62909	.77733	.64256	.76623	1
60	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	0
	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	
′	54°		53°		52°		51°		50°		′

′	40°		41°		42°		43°		44°		′
	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	
0	.64279	.76604	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.64301	.76586	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.64323	.76567	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.64346	.76548	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.64368	.76530	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.64390	.76511	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.64412	.76492	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.64435	.76473	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.64457	.76455	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.64479	.76436	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.64524	.76398	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.64546	.76380	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.64568	.76361	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.64590	.76342	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.64612	.76323	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.64635	.76304	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.64657	.76286	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.64679	.76267	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.64701	.76248	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.64746	.76210	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.64768	.76192	.66083	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.64790	.76173	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.64812	.76154	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.64834	.76135	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.64856	.76116	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.64878	.76097	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.64901	.76078	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.64923	.76059	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.64967	.76022	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.64989	.76003	.66306	.74857	.67602	.73688	.68878	.72497	.70132	.71284	28
33	.65011	.75984	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.65033	.75965	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.65055	.75946	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	.65077	.75927	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.65100	.75908	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.65122	.75889	.66436	.74741	.67730	.73570	.69004	.72377	.70257	.71162	22
39	.65144	.75870	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.65188	.75832	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.65210	.75813	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.65232	.75794	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.65254	.75775	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.65276	.75756	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.65298	.75738	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.65320	.75719	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.65342	.75700	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.65364	.75680	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.65408	.75642	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.65430	.75623	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.65452	.75604	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.65474	.75585	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.65496	.75566	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.65518	.75547	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.65540	.75528	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.65562	.75509	.66870	.74353	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.65584	.75490	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
′	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	Cosin	Sin	′
	49°		48°		47°		46°		45°		

′	0°		1°		2°		3°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	.00000	Infinite.	.01746	57.2900	.03492	28.6363	.05241	19.0811	**60**
1	.00029	3437.75	.01775	56.3506	.03521	28.3994	.05270	18.9755	**59**
2	.00058	1718.87	.01804	55.4415	.03550	28.1664	.05299	18.8711	**58**
3	.00087	1145.92	.01833	54.5613	.03579	27.9372	.05328	18.7678	**57**
4	.00116	859.436	.01862	53.7086	.03609	27.7117	.05357	18.6656	**56**
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	**55**
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	**54**
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	**53**
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	**52**
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	**51**
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	**50**
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	**49**
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	**48**
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	**47**
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	**46**
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	**45**
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	**44**
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	**43**
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	**42**
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	**41**
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	**40**
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	**39**
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	**38**
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	**37**
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	**36**
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	**35**
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	**34**
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	**33**
28	.00815	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	**32**
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	**31**
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	**30**
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	**29**
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	**28**
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	**27**
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	**26**
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	**25**
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	**24**
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	**23**
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	**22**
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	**21**
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	**20**
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	**19**
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	**18**
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	**17**
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	**16**
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	**15**
46	.01338	74.7292	.03084	32.4213	.04833	20.6932	.06584	15.1893	**14**
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	**13**
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	**12**
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	**11**
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	**10**
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	**9**
52	.01513	66.1055	.03259	30.6833	.05007	19.9702	.06759	14.7954	**8**
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	**7**
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	**6**
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	**5**
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	**4**
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	**3**
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	**2**
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	**1**
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	**0**
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	89°		88°		87°		86°		

′	4°		5°		6°		7°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	.06993	14.3007	.08749	11.4301	.10510	9.51436	.12278	8.14435	60
1	.07022	14.2411	.08778	11.3919	.10540	9.48781	.12308	8.12481	59
2	.07051	14.1821	.08807	11.3540	.10569	9.46141	.12338	8.10536	58
3	.07080	14.1235	.08837	11.3163	.10599	9.43515	.12367	8.08600	57
4	.07110	14.0655	.08866	11.2789	.10628	9.40904	.12397	8.06674	56
5	.07139	14.0079	.08895	11.2417	.10657	9.38307	.12426	8.04756	55
6	.07168	13.9507	.08925	11.2048	.10687	9.35724	.12456	8.02848	54
7	.07197	13.8940	.08954	11.1681	.10716	9.33155	.12485	8.00948	53
8	.07227	13.8378	.08983	11.1316	.10746	9.30599	.12515	7.99058	52
9	.07256	13.7821	.09013	11.0954	.10775	9.28058	.12544	7.97176	51
10	.07285	13.7267	.09042	11.0594	.10805	9.25530	.12574	7.95302	50
11	.07314	13.6719	.09071	11.0237	.10834	9.23016	.12603	7.93438	49
12	.07344	13.6174	.09101	10.9882	.10863	9.20516	.12633	7.91582	48
13	.07373	13.5634	.09130	10.9529	.10893	9.18028	.12662	7.89734	47
14	.07402	13.5098	.09159	10.9178	.10922	9.15554	.12692	7.87895	46
15	.07431	13.4566	.09189	10.8829	.10952	9.13093	.12722	7.86064	45
16	.07461	13.4039	.09218	10.8483	.10981	9.10646	.12751	7.84242	44
17	.07490	13.3515	.09247	10.8139	.11011	9.08211	.12781	7.82428	43
18	.07519	13.2996	.09277	10.7797	.11040	9.05789	.12810	7.80622	42
19	.07548	13.2480	.09306	10.7457	.11070	9.03379	.12840	7.78825	41
20	.07578	13.1969	.09335	10.7119	.11099	9.00983	.12869	7.77035	40
21	.07607	13.1461	.09365	10.6783	.11128	8.98598	.12899	7.75254	39
22	.07636	13.0958	.09394	10.6450	.11158	8.96227	.12929	7.73480	38
23	.07665	13.0458	.09423	10.6118	.11187	8.93867	.12958	7.71715	37
24	.07695	12.9962	.09453	10.5789	.11217	8.91520	.12988	7.69957	36
25	.07724	12.9469	.09482	10.5462	.11246	8.89185	.13017	7.68208	35
26	.07753	12.8981	.09511	10.5136	.11276	8.86862	.13047	7.66466	34
27	.07782	12.8496	.09541	10.4813	.11305	8.84551	.13076	7.64732	33
28	.07812	12.8014	.09570	10.4491	.11335	8.82252	.13106	7.63005	32
29	.07841	12.7536	.09600	10.4172	.11364	8.79964	.13136	7.61287	31
30	.07870	12.7062	.09629	10.3854	.11394	8.77689	.13165	7.59575	30
31	.07899	12.6591	.09658	10.3538	.11423	8.75425	.13195	7.57872	29
32	.07929	12.6124	.09688	10.3224	.11452	8.73172	.13224	7.56176	28
33	.07958	12.5660	.09717	10.2913	.11482	8.70931	.13254	7.54487	27
34	.07987	12.5199	.09746	10.2602	.11511	8.68701	.13284	7.52806	26
35	.08017	12.4742	.09776	10.2294	.11541	8.66482	.13313	7.51132	25
36	.08046	12.4288	.09805	10.1988	.11570	8.64275	.13343	7.49465	24
37	.08075	12.3838	.09834	10.1683	.11600	8.62078	.13372	7.47806	23
38	.08104	12.3390	.09864	10.1381	.11629	8.59893	.13402	7.46154	22
39	.08134	12.2946	.09893	10.1080	.11659	8.57718	.13432	7.44509	21
40	.08163	12.2505	.09923	10.0780	.11688	8.55555	.13461	7.42871	20
41	.08192	12.2067	.09952	10.0483	.11718	8.53402	.13491	7.41240	19
42	.08221	12.1632	.09981	10.0187	.11747	8.51259	.13521	7.39616	18
43	.08251	12.1201	.10011	9.98931	.11777	8.49128	.13550	7.37999	17
44	.08280	12.0772	.10040	9.96007	.11806	8.47007	.13580	7.36389	16
45	.08309	12.0346	.10069	9.93101	.11836	8.44896	.13609	7.34786	15
46	.08339	11.9923	.10099	9.90211	.11865	8.42795	.13639	7.33190	14
47	.08368	11.9504	.10128	9.87338	.11895	8.40705	.13669	7.31600	13
48	.08397	11.9087	.10158	9.84482	.11924	8.38625	.13698	7.30018	12
49	.08427	11.8673	.10187	9.81641	.11954	8.36555	.13728	7.28442	11
50	.08456	11.8262	.10216	9.78817	.11983	8.34496	.13758	7.26873	10
51	.08485	11.7853	.10246	8.76009	.12013	8.32446	.13787	7.25310	9
52	.08514	11.7448	.10275	9.73217	.12042	8.30406	.13817	7.23754	8
53	.08544	11.7045	.10305	9.70441	.12072	8.28376	.13846	7.22204	7
54	.08573	11.6645	.10334	9.67680	.12101	8.26355	.13876	7.20661	6
55	.08602	11.6248	.10363	9.64935	.12131	8.24345	.13906	7.19125	5
56	.08632	11.5853	.10393	9.62205	.12160	8.22344	.13935	7.17594	4
57	.08661	11.5461	.10422	9.59490	.12190	8.20352	.13965	7.16071	3
58	.08690	11.5072	.10452	9.56791	.12219	8.18370	.13995	7.14553	2
59	.08720	11.4685	.10481	9.54106	.12249	8.16398	.14024	7.13042	1
60	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	85°		84°		83°		82°		

′	8°		9°		10°		11°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	.14054	7.11537	.15838	6.31375	.17633	5.67128	.19438	5.14455	**60**
1	.14084	7.10038	.15868	6.30189	.17663	5.66165	.19468	5.13658	**59**
2	.14113	7.08546	.15898	6.29007	.17693	5.65205	.19498	5.12862	**58**
3	.14143	7.07059	.15928	6.27829	.17723	5.64248	.19529	5.12069	**57**
4	.14173	7.05579	.15958	6.26655	.17753	5.63295	.19559	5.11279	**56**
5	.14202	7.04105	.15988	6.25486	.17783	5.62344	.19589	5.10490	**55**
6	.14232	7.02637	.16017	6.24321	.17813	5.61397	.19619	5.09704	**54**
7	.14262	7.01174	.16047	6.23160	.17843	5.60452	.19649	5.08921	**53**
8	.14291	6.99718	.16077	6.22003	.17873	5.59511	.19680	5.08139	**52**
9	.14321	6.98268	.16107	6.20851	.17903	5.58573	.19710	5.07360	**51**
10	.14351	6.96823	.16137	6.19703	.17933	5.57638	.19740	5.06584	**50**
11	.14381	6.95385	.16167	6.18559	.17963	5.56706	.19770	5.05809	**49**
12	.14410	6.93952	.16196	6.17419	.17993	5.55777	.19801	5.05037	**48**
13	.14440	6.92525	.16226	6.16283	.18023	5.54851	.19831	5.04267	**47**
14	.14470	6.91104	.16256	6.15151	.18053	5.53927	.19861	5.03499	**46**
15	.14499	6.89688	.16286	6.14023	.18083	5.53007	.19891	5.02734	**45**
16	.14529	6.88278	.16316	6.12899	.18113	5.52090	.19921	5.01971	**44**
17	.14559	6.86874	.16346	6.11779	.18143	5.51176	.19952	5.01210	**43**
18	.14588	6.85475	.16376	6.10664	.18173	5.50264	.19982	5.00451	**42**
19	.14618	6.84082	.16405	6.09552	.18203	5.49356	.20012	4.99695	**41**
20	.14648	6.82694	.16435	6.08444	.18233	5.48451	.20042	4.98940	**40**
21	.14678	6.81312	.16465	6.07340	.18263	5.47548	.20073	4.98188	**39**
22	.14707	6.79936	.16495	6.06240	.18293	5.46648	.20103	4.97438	**38**
23	.14737	6.78564	.16525	6.05143	.18323	5.45751	.20133	4.96690	**37**
24	.14767	6.77199	.16555	6.04051	.18353	5.44857	.20164	4.95945	**36**
25	.14796	6.75838	.16585	6.02962	.18384	5.43966	.20194	4.95201	**35**
26	.14826	6.74483	.16615	6.01878	.18414	5.43077	.20224	4.94460	**34**
27	.14856	6.73133	.16645	6.00797	.18444	5.42192	.20254	4.93721	**33**
28	.14886	6.71789	.16674	5.99720	.18474	5.41309	.20285	4.92984	**32**
29	.14915	6.70450	.16704	5.98646	.18504	5.40429	.20315	4.92249	**31**
30	.14945	6.69116	.16734	5.97576	.18534	5.39552	.20345	4.91516	**30**
31	.14975	6.67787	.16764	5.96510	.18564	5.38677	.20376	4.90785	**29**
32	.15005	6.66463	.16794	5.95448	.18594	5.37805	.20406	4.90056	**28**
33	.15034	6.65144	.16824	5.94390	.18624	5.36936	.20436	4.89330	**27**
34	.15064	6.63831	.16854	5.93335	.18654	5.36070	.20466	4.88605	**26**
35	.15094	6.62523	.16884	5.92283	.18684	5.35206	.20497	4.87882	**25**
36	.15124	6.61219	.16914	5.91236	.18714	5.34345	.20527	4.87162	**24**
37	.15153	6.59921	.16944	5.90191	.18745	5.33487	.20557	4.86444	**23**
38	.15183	6.58627	.16974	5.89151	.18775	5.32631	.20588	4.85727	**22**
39	.15213	6.57339	.17004	5.88114	.18805	5.31778	.20618	4.85013	**21**
40	.15243	6.56055	.17033	5.87080	.18835	5.30928	.20648	4.84300	**20**
41	.15272	6.54777	.17063	5.86051	.18865	5.30080	.20679	4.83590	**19**
42	.15302	6.53503	.17093	5.85024	.18895	5.29235	.20709	4.82882	**18**
43	.15332	6.52234	.17123	5.84001	.18925	5.28393	.20739	4.82175	**17**
44	.15362	6.50970	.17153	5.82982	.18955	5.27553	.20770	4.81471	**16**
45	.15391	6.49710	.17183	5.81966	.18986	5.26715	.20800	4.80769	**15**
46	.15421	6.48456	.17213	5.80953	.19016	5.25880	.20830	4.80068	**14**
47	.15451	6.47206	.17243	5.79944	.19046	5.25048	.20861	4.79370	**13**
48	.15481	6.45961	.17273	5.78938	.19076	5.24218	.20891	4.78673	**12**
49	.15511	6.44720	.17303	5.77936	.19106	5.23391	.20921	4.77978	**11**
50	.15540	6.43484	.17333	5.76937	.19136	5.22566	.20952	4.77286	**10**
51	.15570	6.42253	.17363	5.75941	.19166	5.21744	.20982	4.76595	**9**
52	.15600	6.41026	.17393	5.74949	.19197	5.20925	.21013	4.75906	**8**
53	.15630	6.39804	.17423	5.73960	.19227	5.20107	.21043	4.75219	**7**
54	.15660	6.38587	.17453	5.72974	.19257	5.19293	.21073	4.74534	**6**
55	.15689	6.37374	.17483	5.71992	.19287	5.18480	.21104	4.73851	**5**
56	.15719	6.36165	.17513	5.71013	.19317	5.17671	.21134	4.73170	**4**
57	.15749	6.34961	.17543	5.70037	.19347	5.16863	.21164	4.72490	**3**
58	.15779	6.33761	.17573	5.69064	.19378	5.16058	.21195	4.71813	**2**
59	.15809	6.32566	.17603	5.68094	.19408	5.15256	.21225	4.71137	**1**
60	.15838	6.31375	.17633	5.67128	.19438	5.14455	.21256	4.70463	**0**
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	81°		80°		79°		78°		

′	12° Tan	12° Cotan	13° Tan	13° Cotan	14° Tan	14° Cotan	15° Tan	15° Cotan	′
0	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	**60**
1	.21286	4.69791	.23117	4.32573	.24964	4.00582	.26826	3.72771	**59**
2	.21316	4.69121	.23148	4.32001	.24995	4.00086	.26857	3.72338	**58**
3	.21347	4.68452	.23179	4.31430	.25026	3.99592	.26888	3.71907	**57**
4	.21377	4.67786	.23209	4.30860	.25056	3.99099	.26920	3.71476	**56**
5	.21408	4.67121	.23240	4.30291	.25087	3.98607	.26951	3.71046	**55**
6	.21438	4.66458	.23271	4.29724	.25118	3.98117	.26982	3.70616	**54**
7	.21469	4.65797	.23301	4.29159	.25149	3.97627	.27013	3.70188	**53**
8	.21499	4.65138	.23332	4.28595	.25180	3.97139	.27044	3.69761	**52**
9	.21529	4.64480	.23363	4.28032	.25211	3.96651	.27076	3.69335	**51**
10	.21560	4.63825	.23393	4.27471	.25242	3.96165	.27107	3.68909	**50**
11	.21590	4.63171	.23424	4.26911	.25273	3.95680	.27138	3.68485	**49**
12	.21621	4.62518	.23455	4.26352	.25304	3.95196	.27169	3.68061	**48**
13	.21651	4.61868	.23485	4.25795	.25335	3.94713	.27201	3.67638	**47**
14	.21682	4.61219	.23516	4.25239	.25366	3.94232	.27232	3.67217	**46**
15	.21712	4.60572	.23547	4.24685	.25397	3.93751	.27263	3.66796	**45**
16	.21743	4.59927	.23578	4.24132	.25428	3.93271	.27294	3.66376	**44**
17	.21773	4.59283	.23608	4.23580	.25459	3.92793	.27326	3.65957	**43**
18	.21804	4.58641	.23639	4.23030	.25490	3.92316	.27357	3.65538	**42**
19	.21834	4.58001	.23670	4.22481	.25521	3.91839	.27388	3.65121	**41**
20	.21864	4.57363	.23700	4.21933	.25552	3.91364	.27419	3.64705	**40**
21	.21895	4.56726	.23731	4.21387	.25583	3.90890	.27451	3.64289	**39**
22	.21925	4.56091	.23762	4.20842	.25614	3.90417	.27482	3.63874	**38**
23	.21956	4.55458	.23793	4.20298	.25645	3.89945	.27513	3.63461	**37**
24	.21986	4.54826	.23823	4.19756	.25676	3.89474	.27545	3.63048	**36**
25	.22017	4.54196	.23854	4.19215	.25707	3.89004	.27576	3.62636	**35**
26	.22047	4.53568	.23885	4.18675	.25738	3.88536	.27607	3.62224	**34**
27	.22078	4.52941	.23916	4.18137	.25769	3.88068	.27638	3.61814	**33**
28	.22108	4.52316	.23946	4.17600	.25800	3.87601	.27670	3.61405	**32**
29	.22139	4.51693	.23977	4.17064	.25831	3.87136	.27701	3.60996	**31**
30	.22169	4.51071	.24008	4.16530	.25862	3.86671	.27732	3.60588	**30**
31	.22200	4.50451	.24039	4.15997	.25893	3.86208	.27764	3.60181	**29**
32	.22231	4.49832	.24069	4.15465	.25924	3.85745	.27795	3.59775	**28**
33	.22261	4.49215	.24100	4.14934	.25955	3.85284	.27826	3.59370	**27**
34	.22292	4.48600	.24131	4.14405	.25986	3.84824	.27858	3.58966	**26**
35	.22322	4.47986	.24162	4.13877	.26017	3.84364	.27889	3.58562	**25**
36	.22353	4.47374	.24193	4.13350	.26048	3.83906	.27921	3.58160	**24**
37	.22383	4.46764	.24223	4.12825	.26079	3.83449	.27952	3.57758	**23**
38	.22414	4.46155	.24254	4.12301	.26110	3.82992	.27983	3.57357	**22**
39	.22444	4.45548	.24285	4.11778	.26141	3.82537	.28015	3.56957	**21**
40	.22475	4.44942	.24316	4.11256	.26172	3.82083	.28046	3.56557	**20**
41	.22505	4.44338	.24347	4.10736	.26203	3.81630	.28077	3.56159	**19**
42	.22536	4.43735	.24377	4.10216	.26235	3.81177	.28109	3.55761	**18**
43	.22567	4.43134	.24408	4.09699	.26266	3.80726	.28140	3.55364	**17**
44	.22597	4.42534	.24439	4.09182	.26297	3.80276	.28172	3.54968	**16**
45	.22628	4.41936	.24470	4.08666	.26328	3.79827	.28203	3.54573	**15**
46	.22658	4.41340	.24501	4.08152	.26359	3.79378	.28234	3.54179	**14**
47	.22689	4.40745	.24532	4.07639	.26390	3.78931	.28266	3.53785	**13**
48	.22719	4.40152	.24562	4.07127	.26421	3.78485	.28297	3.53393	**12**
49	.22750	4.39560	.24593	4.06616	.26452	3.78040	.28329	3.53001	**11**
50	.22781	4.38969	.24624	4.06107	.26483	3.77595	.28360	3.52609	**10**
51	.22811	4.38381	.24655	4.05599	.26515	3.77152	.28391	3.52219	**9**
52	.22842	4.37793	.24686	4.05092	.26546	3.76709	.28423	3.51829	**8**
53	.22872	4.37207	.24717	4.04586	.26577	3.76268	.28454	3.51441	**7**
54	.22903	4.36623	.24747	4.04081	.26608	3.75828	.28486	3.51053	**6**
55	.22934	4.36040	.24778	4.03578	.26639	3.75388	.28517	3.50666	**5**
56	.22964	4.35459	.24809	4.03076	.26670	3.74950	.28549	3.50279	**4**
57	.22995	4.34879	.24840	4.02574	.26701	3.74512	.28580	3.49894	**3**
58	.23026	4.34300	.24871	4.02074	.26733	3.74075	.28612	3.49509	**2**
59	.23056	4.33723	.24902	4.01576	.26764	3.73640	.28643	3.49125	**1**
60	.23087	4.33148	.24933	4.01078	.26795	3.73205	.28675	3.48741	**0**
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	77°		76°		75°		74°		

′	16°		17°		18°		19°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2.87700	50
11	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35020	2.85555	42
19	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.29337	3.40869	.31242	3.20079	.33169	3.01489	.35118	2.84758	39
22	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.29843	3.35087	.31754	3.14922	.33686	2.96858	.35641	2.80574	23
38	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.29906	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.30097	3.32264	.32010	3.12400	.33945	2.94591	.35904	2.78523	15
46	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.30224	3.30868	.32139	3.11153	.34075	2.93468	.36035	2.77507	11
50	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	73°		72°		71°		70°		

′	20° Tan	20° Cotan	21° Tan	21° Cotan	22° Tan	22° Cotan	23° Tan	23° Cotan	′
0	.36397	2.74748	.38386	2.60509	.40403	2.47509	.42447	2.35585	60
1	.36430	2.74499	.38420	2.60283	.40436	2.47302	.42482	2.35395	59
2	.36463	2.74251	.38453	2.60057	.40470	2.47095	.42516	2.35205	58
3	.36496	2.74004	.38487	2.59831	.40504	2.46888	.42551	2.35015	57
4	.36529	2.73756	.38520	2.59606	.40538	2.46682	.42585	2.34825	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	55
6	.36595	2.73263	.38587	2.59156	.40606	2.46270	.42654	2.34447	54
7	.36628	2.73017	.38620	2.58932	.40640	2.46065	.42688	2.34258	53
8	.36661	2.72771	.38654	2.58708	.40674	2.45860	.42722	2.34069	52
9	.36694	2.72526	.38687	2.58484	.40707	2.45655	.42757	2.33881	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	50
11	.36760	2.72036	.38754	2.58038	.40775	2.45246	.42826	2.33505	49
12	.36793	2.71792	.38787	2.57815	.40809	2.45043	.42860	2.33317	48
13	.36826	2.71548	.38821	2.57593	.40843	2.44839	.42894	2.33130	47
14	.36859	2.71305	.38854	2.57371	.40877	2.44636	.42929	2.32943	46
15	.36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	45
16	.36925	2.70819	.38921	2.56928	.40945	2.44230	.42998	2.32570	44
17	.36958	2.70577	.38955	2.56707	.40979	2.44027	.43032	2.32383	43
18	.36991	2.70335	.38988	2.56487	.41013	2.43825	.43067	2.32197	42
19	.37024	2.70094	.39022	2.56266	.41047	2.43623	.43101	2.32012	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	40
21	.37090	2.69612	.39089	2.55827	.41115	2.43220	.43170	2.31641	39
22	.37123	2.69371	.39122	2.55608	.41149	2.43019	.43205	2.31456	38
23	.37157	2.69131	.39156	2.55389	.41183	2.42819	.43239	2.31271	37
24	.37190	2.68892	.39190	2.55170	.41217	2.42618	.43274	2.31086	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	35
26	.37256	2.68414	.39257	2.54734	.41285	2.42218	.43343	2.30718	34
27	.37289	2.68175	.39290	2.54516	.41319	2.42019	.43378	2.30534	33
28	.37322	2.67937	.39324	2.54299	.41353	2.41819	.43412	2.30351	32
29	.37355	2.67700	.39357	2.54082	.41387	2.41620	.43447	2.30167	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	30
31	.37422	2.67225	.39425	2.53648	.41455	2.41223	.43516	2.29801	29
32	.37455	2.66989	.39458	2.53432	.41490	2.41025	.43550	2.29619	28
33	.37488	2.66752	.39492	2.53217	.41524	2.40827	.43585	2.29437	27
34	.37521	2.66516	.39526	2.53001	.41558	2.40629	.43620	2.29254	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	25
36	.37588	2.66046	.39593	2.52571	.41626	2.40235	.43689	2.28891	24
37	.37621	2.65811	.39626	2.52357	.41660	2.40038	.43724	2.28710	23
38	.37654	2.65576	.39660	2.52142	.41694	2.39841	.43758	2.28528	22
39	.37687	2.65342	.39694	2.51929	.41728	2.39645	.43793	2.28348	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	20
41	.37754	2.64875	.39761	2.51502	.41797	2.39253	.43862	2.27987	19
42	.37787	2.64642	.39795	2.51289	.41831	2.39058	.43897	2.27806	18
43	.37820	2.64410	.39829	2.51076	.41865	2.38863	.43932	2.27626	17
44	.37853	2.64177	.39862	2.50864	.41899	2.38668	.43966	2.27447	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	15
46	.37920	2.63714	.39930	2.50440	.41968	2.38279	.44036	2.27088	14
47	.37953	2.63483	.39963	2.50229	.42002	2.38084	.44071	2.26909	13
48	.37986	2.63252	.39997	2.50018	.42036	2.37891	.44105	2.26730	12
49	.38020	2.63021	.40031	2.49807	.42070	2.37697	.44140	2.26552	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	10
51	.38086	2.62561	.40098	2.49386	.42139	2.37311	.44210	2.26196	9
52	.38120	2.62332	.40132	2.49177	.42173	2.37118	.44244	2.26018	8
53	.38153	2.62103	.40166	2.48967	.42207	2.36925	.44279	2.25840	7
54	.38186	2.61874	.40200	2.48758	.42242	2.36733	.44314	2.25663	6
55	.38220	2.61646	.40234	2.48549	.42276	2.36541	.44349	2.25486	5
56	.38253	2.61418	.40267	2.48340	.42310	2.36349	.44384	2.25309	4
57	.38286	2.61190	.40301	2.48132	.42345	2.36158	.44418	2.25132	3
58	.38320	2.60963	.40335	2.47924	.42379	2.35967	.44453	2.24956	2
59	.38353	2.60736	.40369	2.47716	.42413	2.35776	.44488	2.24780	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	0
′	69° Cotan	69° Tan	68° Cotan	68° Tan	67° Cotan	67° Tan	66° Cotan	66° Tan	′

′	24°		25°		26°		27°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	.44523	2.24604	.46631	2.14451	.48773	2.05030	.50953	1.96261	60
1	.44558	2.24428	.46666	2.14288	.48809	2.04879	.50989	1.96120	59
2	.44593	2.24252	.46702	2.14125	.48845	2.04728	.51026	1.95979	58
3	.44627	2.24077	.46737	2.13963	.48881	2.04577	.51063	1.95838	57
4	.44662	2.23902	.46772	2.13801	.48917	2.04426	.51099	1.95698	56
5	.44697	2.23727	.46808	2.13639	.48953	2.04276	.51136	1.95557	55
6	.44732	2.23553	.46843	2.13477	.48989	2.04125	.51173	1.95417	54
7	.44767	2.23378	.46879	2.13316	.49026	2.03975	.51209	1.95277	53
8	.44802	2.23204	.46914	2.13154	.49062	2.03825	.51246	1.95137	52
9	.44837	2.23030	.46950	2.12993	.49098	2.03675	.51283	1.94997	51
10	.44872	2.22857	.46985	2.12832	.49134	2.03526	.51319	1.94858	50
11	.44907	2.22683	.47021	2.12671	.49170	2.03376	.51356	1.94718	49
12	.44942	2.22510	.47056	2.12511	.49206	2.03227	.51393	1.94579	48
13	.44977	2.22337	.47092	2.12350	.49242	2.03078	.51430	1.94440	47
14	.45012	2.22164	.47128	2.12190	.49278	2.02929	.51467	1.94301	46
15	.45047	2.21992	.47163	2.12030	.49315	2.02780	.51503	1.94162	45
16	.45082	2.21819	.47199	2.11871	.49351	2.02631	.51540	1.94023	44
17	.45117	2.21647	.47234	2.11711	.49387	2.02483	.51577	1.93885	43
18	.45152	2.21475	.47270	2.11552	.49423	2.02335	.51614	1.93746	42
19	.45187	2.21304	.47305	2.11392	.49459	2.02187	.51651	1.93608	41
20	.45222	2.21132	.47341	2.11233	.49495	2.02039	.51688	1.93470	40
21	.45257	2.20961	.47377	2.11075	.49532	2.01891	.51724	1.93332	39
22	.45292	2.20790	.47412	2.10916	.49568	2.01743	.51761	1.93195	38
23	.45327	2.20619	.47448	2.10758	.49604	2.01596	.51798	1.93057	37
24	.45362	2.20449	.47483	2.10600	.49640	2.01449	.51835	1.92920	36
25	.45397	2.20278	.47519	2.10442	.49677	2.01302	.51872	1.92782	35
26	.45432	2.20108	.47555	2.10284	.49713	2.01155	.51909	1.92645	34
27	.45467	2.19938	.47590	2.10126	.49749	2.01008	.51946	1.92508	33
28	.45502	2.19769	.47626	2.09969	.49786	2.00862	.51983	1.92371	32
29	.45538	2.19599	.47662	2.09811	.49822	2.00715	.52020	1.92235	31
30	.45573	2.19430	.47698	2.09654	.49858	2.00569	.52057	1.92098	30
31	.45608	2.19261	.47733	2.09498	.49894	2.00423	.52094	1.91962	29
32	.45643	2.19092	.47769	2.09341	.49931	2.00277	.52131	1.91826	28
33	.45678	2.18923	.47805	2.09184	.49967	2.00131	.52168	1.91690	27
34	.45713	2.18755	.47840	2.09028	.50004	1.99986	.52205	1.91554	26
35	.45748	2.18587	.47876	2.08872	.50040	1.99841	.52242	1.91418	25
36	.45784	2.18419	.47912	2.08716	.50076	1.99695	.52279	1.91282	24
37	.45819	2.18251	.47948	2.08560	.50113	1.99550	.52316	1.91147	23
38	.45854	2.18084	.47984	2.08405	.50149	1.99406	.52353	1.91012	22
39	.45889	2.17916	.48019	2.08250	.50185	1.99261	.52390	1.90876	21
40	.45924	2.17749	.48055	2.08094	.50222	1.99116	.52427	1.90741	20
41	.45960	2.17582	.48091	2.07939	.50258	1.98972	.52464	1.90607	19
42	.45995	2.17416	.48127	2.07785	.50295	1.98828	.52501	1.90472	18
43	.46030	2.17249	.48163	2.07630	.50331	1.98684	.52538	1.90337	17
44	.46065	2.17083	.48198	2.07476	.50368	1.98540	.52575	1.90203	16
45	.46101	2.16917	.48234	2.07321	.50404	1.98396	.52613	1.90069	15
46	.46136	2.16751	.48270	2.07167	.50441	1.98253	.52650	1.89935	14
47	.46171	2.16585	.48306	2.07014	.50477	1.98110	.52687	1.89801	13
48	.46206	2.16420	.48342	2.06860	.50514	1.97966	.52724	1.89667	12
49	.46242	2.16255	.48378	2.06706	.50550	1.97823	.52761	1.89533	11
50	.46277	2.16090	.48414	2.06553	.50587	1.97681	.52798	1.89400	10
51	.46312	2.15925	.48450	2.06400	.50623	1.97538	.52836	1.89266	9
52	.46348	2.15760	.48486	2.06247	.50660	1.97395	.52873	1.89133	8
53	.46383	2.15596	.48521	2.06094	.50696	1.97253	.52910	1.89000	7
54	.46418	2.15432	.48557	2.05942	.50733	1.97111	.52947	1.88867	6
55	.46454	2.15268	.48593	2.05790	.50769	1.96969	.52985	1.88734	5
56	.46489	2.15104	.48629	2.05637	.50806	1.96827	.53022	1.88602	4
57	.46525	2.14940	.48665	2.05485	.50843	1.96685	.53059	1.88469	3
58	.46560	2.14777	.48701	2.05333	.50879	1.96544	.53096	1.88337	2
59	.46595	2.14614	.48737	2.05182	.50916	1.96402	.53134	1.88205	1
60	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	65°		64°		63°		62°		

′	28°		29°		30°		31°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	.53171	1.88073	.55431	1.80405	.57735	1.73205	.60086	1.66428	60
1	.53208	1.87941	.55469	1.80281	.57774	1.73089	.60126	1.66318	59
2	.53246	1.87809	.55507	1.80158	.57813	1.72973	.60165	1.66209	58
3	.53283	1.87677	.55545	1.80034	.57851	1.72857	.60205	1.66099	57
4	.53320	1.87546	.55583	1.79911	.57890	1.72741	.60245	1.65990	56
5	.53358	1.87415	.55621	1.79788	.57929	1.72625	.60284	1.65881	55
6	.53395	1.87283	.55659	1.79665	.57968	1.72509	.60324	1.65772	54
7	.53432	1.87152	.55697	1.79542	.58007	1.72393	.60364	1.65663	53
8	.53470	1.87021	.55736	1.79419	.58046	1.72278	.60403	1.65554	52
9	.53507	1.86891	.55774	1.79296	.58085	1.72163	.60443	1.65445	51
10	.53545	1.86760	.55812	1.79174	.58124	1.72047	.60483	1.65337	50
11	.53582	1.86630	.55850	1.79051	.58162	1.71932	.60522	1.65228	49
12	.53620	1.86499	.55888	1.78929	.58201	1.71817	.60562	1.65120	48
13	.53657	1.86369	.55926	1.78807	.58240	1.71702	.60602	1.65011	47
14	.53694	1.86239	.55964	1.78685	.58279	1.71588	.60642	1.64903	46
15	.53732	1.86109	.56003	1.78563	.58318	1.71473	.60681	1.64795	45
16	.53769	1.85979	.56041	1.78441	.58357	1.71358	.60721	1.64687	44
17	.53807	1.85850	.56079	1.78319	.58396	1.71244	.60761	1.64579	43
18	.53844	1.85720	.56117	1.78198	.58435	1.71129	.60801	1.64471	42
19	.53882	1.85591	.56156	1.78077	.58474	1.71015	.60841	1.64363	41
20	.53920	1.85462	.56194	1.77955	.58513	1.70901	.60881	1.64256	40
21	.53957	1.85333	.56232	1.77834	.58552	1.70787	.60921	1.64148	39
22	.53995	1.85204	.56270	1.77713	.58591	1.70673	.60960	1.64041	38
23	.54032	1.85075	.56309	1.77592	.58631	1.70560	.61000	1.63934	37
24	.54070	1.84946	.56347	1.77471	.58670	1.70446	.61040	1.63826	36
25	.54107	1.84818	.56385	1.77351	.58709	1.70332	.61080	1.63719	35
26	.54145	1.84689	.56424	1.77230	.58748	1.70219	.61120	1.63612	34
27	.54183	1.84561	.56462	1.77110	.58787	1.70106	.61160	1.63505	33
28	.54220	1.84433	.56501	1.76990	.58826	1.69992	.61200	1.63398	32
29	.54258	1.84305	.56539	1.76869	.58865	1.69879	.61240	1.63292	31
30	.54296	1.84177	.56577	1.76749	.58905	1.69766	.61280	1.63185	30
31	.54333	1.84049	.56616	1.76629	.58944	1.69653	.61320	1.63079	29
32	.54371	1.83922	.56654	1.76510	.58983	1.69541	.61360	1.62972	28
33	.54409	1.83794	.56693	1.76390	.59022	1.69428	.61400	1.62866	27
34	.54446	1.83667	.56731	1.76271	.59061	1.69316	.61440	1.62760	26
35	.54484	1.83540	.56769	1.76151	.59101	1.69203	.61480	1.62654	25
36	.54522	1.83413	.56808	1.76032	.59140	1.69091	.61520	1.62548	24
37	.54560	1.83286	.56846	1.75913	.59179	1.68979	.61561	1.62442	23
38	.54597	1.83159	.56885	1.75794	.59218	1.68866	.61601	1.62336	22
39	.54635	1.83033	.56923	1.75675	.59258	1.68754	.61641	1.62230	21
40	.54673	1.82906	.56962	1.75556	.59297	1.68643	.61681	1.62125	20
41	.54711	1.82780	.57000	1.75437	.59336	1.68531	.61721	1.62019	19
42	.54748	1.82654	.57039	1.75319	.59376	1.68419	.61761	1.61914	18
43	.54786	1.82528	.57078	1.75200	.59415	1.68308	.61801	1.61808	17
44	.54824	1.82402	.57116	1.75082	.59454	1.68196	.61842	1.61703	16
45	.54862	1.82276	.57155	1.74964	.59494	1.68085	.61882	1.61598	15
46	.54900	1.82150	.57193	1.74846	.59533	1.67974	.61922	1.61493	14
47	.54938	1.82025	.57232	1.74728	.59573	1.67863	.61962	1.61388	13
48	.54975	1.81899	.57271	1.74610	.59612	1.67752	.62003	1.61283	12
49	.55013	1.81774	.57309	1.74492	.59651	1.67641	.62043	1.61179	11
50	.55051	1.81649	.57348	1.74375	.59691	1.67530	.62083	1.61074	10
51	.55089	1.81524	.57386	1.74257	.59730	1.67419	.62124	1.60970	9
52	.55127	1.81399	.57425	1.74140	.59770	1.67309	.62164	1.60865	8
53	.55165	1.81274	.57464	1.74022	.59809	1.67198	.62204	1.60761	7
54	.55203	1.81150	.57503	1.73905	.59849	1.67088	.62245	1.60657	6
55	.55241	1.81025	.57541	1.73788	.59888	1.66978	.62285	1.60553	5
56	.55279	1.80901	.57580	1.73671	.59928	1.66867	.62325	1.60449	4
57	.55317	1.80777	.57619	1.73555	.59967	1.66757	.62366	1.60345	3
58	.55355	1.80653	.57657	1.73438	.60007	1.66647	.62406	1.60241	2
59	.55393	1.80529	.57696	1.73321	.60046	1.66538	.62446	1.60137	1
60	.55431	1.80405	.57735	1.73205	.60086	1.66428	.62487	1.60033	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	61°		60°		59°		58°		

′	32°		33°		34°		35°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	60
1	.62527	1.59930	.64982	1.53888	.67493	1.48163	.70064	1.42726	59
2	.62568	1.59826	.65024	1.53791	.67536	1.48070	.70107	1.42638	58
3	.62608	1.59723	.65065	1.53693	.67578	1.47977	.70151	1.42550	57
4	.62649	1.59620	.65106	1.53595	.67620	1.47885	.70194	1.42462	56
5	.62689	1.59517	.65148	1.53497	.67663	1.47792	.70238	1.42374	55
6	.62730	1.59414	.65189	1.53400	.67705	1.47699	.70281	1.42286	54
7	.62770	1.59311	.65231	1.53302	.67748	1.47607	.70325	1.42198	53
8	.62811	1.59208	.65272	1.53205	.67790	1.47514	.70368	1.42110	52
9	.62852	1.59105	.65314	1.53107	.67832	1.47422	.70412	1.42022	51
10	.62892	1.59002	.65355	1.53010	.67875	1.47330	.70455	1.41934	50
11	.62933	1.58900	.65397	1.52913	.67917	1.47238	.70499	1.41847	49
12	.62973	1.58797	.65438	1.52816	.67960	1.47146	.70542	1.41759	48
13	.63014	1.58695	.65480	1.52719	.68002	1.47053	.70586	1.41672	47
14	.63055	1.58593	.65521	1.52622	.68045	1.46962	.70629	1.41584	46
15	.63095	1.58490	.65563	1.52525	.68088	1.46870	.70673	1.41497	45
16	.63136	1.58388	.65604	1.52429	.68130	1.46778	.70717	1.41409	44
17	.63177	1.58286	.65646	1.52332	.68173	1.46686	.70760	1.41322	43
18	.63217	1.58184	.65688	1.52235	.68215	1.46595	.70804	1.41235	42
19	.63258	1.58083	.65729	1.52139	.68258	1.46503	.70848	1.41148	41
20	.63299	1.57981	.65771	1.52043	.68301	1.46411	.70891	1.41061	40
21	.63340	1.57879	.65813	1.51946	.68343	1.46320	.70935	1.40974	39
22	.63380	1.57778	.65854	1.51850	.68386	1.46229	.70979	1.40887	38
23	.63421	1.57676	.65896	1.51754	.68429	1.46137	.71023	1.40800	37
24	.63462	1.57575	.65938	1.51658	.68471	1.46046	.71066	1.40714	36
25	.63503	1.57474	.65980	1.51562	.68514	1.45955	.71110	1.40627	35
26	.63544	1.57372	.66021	1.51466	.68557	1.45864	.71154	1.40540	34
27	.63584	1.57271	.66063	1.51370	.68600	1.45773	.71198	1.40454	33
28	.63625	1.57170	.66105	1.51275	.68642	1.45682	.71242	1.40367	32
29	.63666	1.57069	.66147	1.51179	.68685	1.45592	.71285	1.40281	31
30	.63707	1.56969	.66189	1.51084	.68728	1.45501	.71329	1.40195	30
31	.63748	1.56868	.66230	1.50988	.68771	1.45410	.71373	1.40109	29
32	.63789	1.56767	.66272	1.50893	.68814	1.45320	.71417	1.40022	28
33	.63830	1.56667	.66314	1.50797	.68857	1.45229	.71461	1.39936	27
34	.63871	1.56566	.66356	1.50702	.68900	1.45139	.71505	1.39850	26
35	.63912	1.56466	.66398	1.50607	.68942	1.45049	.71549	1.39764	25
36	.63953	1.56366	.66440	1.50512	.68985	1.44958	.71593	1.39679	24
37	.63994	1.56265	.66482	1.50417	.69028	1.44868	.71637	1.39593	23
38	.64035	1.56165	.66524	1.50322	.69071	1.44778	.71681	1.39507	22
39	.64076	1.56065	.66566	1.50228	.69114	1.44688	.71725	1.39421	21
40	.64117	1.55966	.66608	1.50133	.69157	1.44598	.71769	1.39336	20
41	.64158	1.55866	.66650	1.50038	.69200	1.44508	.71813	1.39250	19
42	.64199	1.55766	.66692	1.49944	.69243	1.44418	.71857	1.39165	18
43	.64240	1.55666	.66734	1.49849	.69286	1.44329	.71901	1.39079	17
44	.64281	1.55567	.66776	1.49755	.69329	1.44239	.71946	1.38994	16
45	.64322	1.55467	.66818	1.49661	.69372	1.44149	.71990	1.38909	15
46	.64363	1.55368	.66860	1.49566	.69416	1.44060	.72034	1.38824	14
47	.64404	1.55269	.66902	1.49472	.69459	1.43970	.72078	1.38738	13
48	.64446	1.55170	.66944	1.49378	.69502	1.43881	.72122	1.38653	12
49	.64487	1.55071	.66986	1.49284	.69545	1.43792	.72167	1.38568	11
50	.64528	1.54972	.67028	1.49190	.69588	1.43703	.72211	1.38484	10
51	.64569	1.54873	.67071	1.49097	.69631	1.43614	.72255	1.38399	9
52	.64610	1.54774	.67113	1.49003	.69675	1.43525	.72299	1.38314	8
53	.64652	1.54675	.67155	1.48909	.69718	1.43436	.72344	1.38229	7
54	.64693	1.54576	.67197	1.48816	.69761	1.43347	.72388	1.38145	6
55	.64734	1.54478	.67239	1.48722	.69804	1.43258	.72432	1.38060	5
56	.64775	1.54379	.67282	1.48629	.69847	1.43169	.72477	1.37976	4
57	.64817	1.54281	.67324	1.48536	.69891	1.43080	.72521	1.37891	3
58	.64858	1.54183	.67366	1.48442	.69934	1.42992	.72565	1.37807	2
59	.64899	1.54085	.67409	1.48349	.69977	1.42903	.72610	1.37722	1
60	.64941	1.53986	.67451	1.48256	.70021	1.42815	.72654	1.37638	0
	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	
′	57°		56°		55°		54°		′

′	36° Tan	36° Cotan	37° Tan	37° Cotan	38° Tan	38° Cotan	39° Tan	39° Cotan	′
0	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	**60**
1	.72699	1.37554	.75401	1.32624	.78175	1.27917	.81027	1.23416	**59**
2	.72743	1.37470	.75447	1.32544	.78222	1.27841	.81075	1.23343	**58**
3	.72788	1.37386	.75492	1.32464	.78269	1.27764	.81123	1.23270	**57**
4	.72832	1.37302	.75538	1.32384	.78316	1.27688	.81171	1.23196	**56**
5	.72877	1.37218	.75584	1.32304	.78363	1.27611	.81220	1.23123	**55**
6	.72921	1.37134	.75629	1.32224	.78410	1.27535	.81268	1.23050	**54**
7	.72966	1.37050	.75675	1.32144	.78457	1.27458	.81316	1.22977	**53**
8	.73010	1.36967	.75721	1.32064	.78504	1.27382	.81364	1.22904	**52**
9	.73055	1.36883	.75767	1.31984	.78551	1.27306	.81413	1.22831	**51**
10	.73100	1.36800	.75812	1.31904	.78598	1.27230	.81461	1.22758	**50**
11	.73144	1.36716	.75858	1.31825	.78645	1.27153	.81510	1.22685	**49**
12	.73189	1.36633	.75904	1.31745	.78692	1.27077	.81558	1.22612	**48**
13	.73234	1.36549	.75950	1.31666	.78739	1.27001	.81606	1.22539	**47**
14	.73278	1.36466	.75996	1.31586	.78786	1.26925	.81655	1.22467	**46**
15	.73323	1.36383	.76042	1.31507	.78834	1.26849	.81703	1.22394	**45**
16	.73368	1.36300	.76088	1.31427	.78881	1.26774	.81752	1.22321	**44**
17	.73413	1.36217	.76134	1.31348	.78928	1.26698	.81800	1.22249	**43**
18	.73457	1.36134	.76180	1.31269	.78975	1.26622	.81849	1.22176	**42**
19	.73502	1.36051	.76226	1.31190	.79022	1.26546	.81898	1.22104	**41**
20	.73547	1.35968	.76272	1.31110	.79070	1.26471	.81946	1.22031	**40**
21	.73592	1.35885	.76318	1.31031	.79117	1.26395	.81995	1.21959	**39**
22	.73637	1.35802	.76364	1.30952	.79164	1.26319	.82044	1.21886	**38**
23	.73681	1.35719	.76410	1.30873	.79212	1.26244	.82092	1.21814	**37**
24	.73726	1.35637	.76456	1.30795	.79259	1.26169	.82141	1.21742	**36**
25	.73771	1.35554	.76502	1.30716	.79306	1.26093	.82190	1.21670	**35**
26	.73816	1.35472	.76548	1.30637	.79354	1.26018	.82238	1.21598	**34**
27	.73861	1.35389	.76594	1.30558	.79401	1.25943	.82287	1.21526	**33**
28	.73906	1.35307	.76640	1.30480	.79449	1.25867	.82336	1.21454	**32**
29	.73951	1.35224	.76686	1.30401	.79496	1.25792	.82385	1.21382	**31**
30	.73996	1.35142	.76733	1.30323	.79544	1.25717	.82434	1.21310	**30**
31	.74041	1.35060	.76779	1.30244	.79591	1.25642	.82483	1.21238	**29**
32	.74086	1.34978	.76825	1.30166	.79639	1.25567	.82531	1.21166	**28**
33	.74131	1.34896	.76871	1.30087	.79686	1.25492	.82580	1.21094	**27**
34	.74176	1.34814	.76918	1.30009	.79734	1.25417	.82629	1.21023	**26**
35	.74221	1.34732	.76964	1.29931	.79781	1.25343	.82678	1.20951	**25**
36	.74267	1.34650	.77010	1.29853	.79829	1.25268	.82727	1.20879	**24**
37	.74312	1.34568	.77057	1.29775	.79877	1.25193	.82776	1.20808	**23**
38	.74357	1.34487	.77103	1.29696	.79924	1.25118	.82825	1.20736	**22**
39	.74402	1.34405	.77149	1.29618	.79972	1.25044	.82874	1.20665	**21**
40	.74447	1.34323	.77196	1.29541	.80020	1.24969	.82923	1.20593	**20**
41	.74492	1.34242	.77242	1.29463	.80067	1.24895	.82972	1.20522	**19**
42	.74538	1.34160	.77289	1.29385	.80115	1.24820	.83022	1.20451	**18**
43	.74583	1.34079	.77335	1.29307	.80163	1.24746	.83071	1.20379	**17**
44	.74628	1.33998	.77382	1.29229	.80211	1.24672	.83120	1.20308	**16**
45	.74674	1.33916	.77428	1.29152	.80258	1.24597	.83169	1.20237	**15**
46	.74719	1.33835	.77475	1.29074	.80306	1.24523	.83218	1.20166	**14**
47	.74764	1.33754	.77521	1.28997	.80354	1.24449	.83268	1.20095	**13**
48	.74810	1.33673	.77568	1.28919	.80402	1.24375	.83317	1.20024	**12**
49	.74855	1.33592	.77615	1.28842	.80450	1.24301	.83366	1.19953	**11**
50	.74900	1.33511	.77661	1.28764	.80498	1.24227	.83415	1.19882	**10**
51	.74946	1.33430	.77708	1.28687	.80546	1.24153	.83465	1.19811	**9**
52	.74991	1.33349	.77754	1.28610	.80594	1.24079	.83514	1.19740	**8**
53	.75037	1.33268	.77801	1.28533	.80642	1.24005	.83564	1.19669	**7**
54	.75082	1.33187	.77848	1.28456	.80690	1.23931	.83613	1.19599	**6**
55	.75128	1.33107	.77895	1.28379	.80738	1.23858	.83662	1.19528	**5**
56	.75173	1.33026	.77941	1.28302	.80786	1.23784	.83712	1.19457	**4**
57	.75219	1.32946	.77988	1.28225	.80834	1.23710	.83761	1.19387	**3**
58	.75264	1.32865	.78035	1.28148	.80882	1.23637	.83811	1.19316	**2**
59	.75310	1.32785	.78082	1.28071	.80930	1.23563	.83860	1.19246	**1**
60	.75355	1.32704	.78129	1.27994	.80978	1.23490	.83910	1.19175	**0**
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	53°		52°		51°		50°		

′	40°		41°		42°		43°		′
	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	
0	.83910	1.19175	.86929	1.15037	.90040	1.11061	.93252	1.07237	60
1	.83960	1.19105	.86980	1.14969	.90093	1.10996	.93306	1.07174	59
2	.84009	1.19035	.87031	1.14902	.90146	1.10931	.93360	1.07112	58
3	.84059	1.18964	.87082	1.14834	.90199	1.10867	.93415	1.07049	57
4	.84108	1.18894	.87133	1.14767	.90251	1.10802	.93469	1.06987	56
5	.85158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	55
6	.84208	1.18754	.87236	1.14632	.90357	1.10672	.93578	1.06862	54
7	.84258	1.18684	.87287	1.14565	.90410	1.10607	.93633	1.06800	53
8	.84307	1.18614	.87338	1.14498	.90463	1.10543	.93688	1.06738	52
9	.84357	1.18544	.87389	1.14430	.90516	1.10478	.93742	1.06676	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	50
11	.84457	1.18404	.87492	1.14296	.90621	1.10349	.93852	1.06551	49
12	.84507	1.18334	.87543	1.14229	.90674	1.10285	.93906	1.06489	48
13	.84556	1.18264	.87595	1.14162	.90727	1.10220	.93961	1.06427	47
14	.84606	1.18194	.87646	1.14095	.90781	1.10156	.94016	1.06365	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	45
16	.84706	1.18055	.87749	1.13961	.90887	1.10027	.94125	1.06241	44
17	.84756	1.17986	.87801	1.13894	.90940	1.09963	.94180	1.06179	43
18	.84806	1.17916	.87852	1.13828	.90993	1.09899	.94235	1.06117	42
19	.84856	1.17846	.87904	1.13761	.91046	1.09834	.94290	1.06056	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	40
21	.84956	1.17708	.88007	1.13627	.91153	1.09706	.94400	1.05932	39
22	.85006	1.17638	.88059	1.13561	.91206	1.09642	.94455	1.05870	38
23	.85057	1.17569	.88110	1.13494	.91259	1.09578	.94510	1.05809	37
24	.85107	1.17500	.88162	1.13428	.91313	1.09514	.94565	1.05747	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	35
26	.85207	1.17361	.88265	1.13295	.91419	1.09386	.94676	1.05624	34
27	.85257	1.17292	.88317	1.13228	.91473	1.09322	.94731	1.05562	33
28	.85308	1.17223	.88369	1.13162	.91526	1.09258	.94786	1.05501	32
29	.85358	1.17154	.88421	1.13096	.91580	1.09195	.94841	1.05439	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	30
31	.85458	1.17016	.88524	1.12963	.91687	1.09067	.94952	1.05317	29
32	.85509	1.16947	.88576	1.12897	.91740	1.09003	.95007	1.05255	28
33	.85559	1.16878	.88628	1.12831	.91794	1.08940	.95062	1.05194	27
34	.85609	1.16809	.88680	1.12765	.91847	1.08876	.95118	1.05133	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	25
36	.85710	1.16672	.88784	1.12633	.91955	1.08749	.95229	1.05010	24
37	.85761	1.16603	.88836	1.12567	.92008	1.08686	.95284	1.04949	23
38	.85811	1.16535	.88888	1.12501	.92062	1.08622	.95340	1.04888	22
39	.85862	1.16466	.88940	1.12435	.92116	1.08559	.95395	1.04827	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	20
41	.85963	1.16329	.89045	1.12303	.92224	1.08432	.95506	1.04705	19
42	.86014	1.16261	.89097	1.12238	.92277	1.08369	.95562	1.04644	18
43	.86064	1.16192	.89149	1.12172	.92331	1.08306	.95618	1.04583	17
44	.86115	1.16124	.89201	1.12106	.92385	1.08243	.95673	1.04522	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	15
46	.86216	1.15987	.89306	1.11975	.92493	1.08116	.95785	1.04401	14
47	.86267	1.15919	.89358	1.11909	.92547	1.08053	.95841	1.04340	13
48	.86318	1.15851	.89410	1.11844	.92601	1.07990	.95897	1.04279	12
49	.86368	1.15783	.89463	1.11778	.92655	1.07927	.95952	1.04218	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.04158	10
51	.86470	1.15647	.89567	1.11648	.92763	1.07801	.96064	1.04097	9
52	.86521	1.15579	.89620	1.11582	.92817	1.07738	.96120	1.04036	8
53	.86572	1.15511	.89672	1.11517	.92872	1.07676	.96176	1.03976	7
54	.86623	1.15443	.89725	1.11452	.92926	1.07613	.96232	1.03915	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	5
56	.86725	1.15308	.89830	1.11321	.93034	1.07487	.96344	1.03794	4
57	.86776	1.15240	.89883	1.11256	.93088	1.07425	.96400	1.03734	3
58	.86827	1.15172	.89935	1.11191	.93143	1.07362	.96457	1.03674	2
59	.86878	1.15104	.89988	1.11126	.93197	1.07299	.96513	1.03613	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	0
′	Cotan	Tan	Cotan	Tan	Cotan	Tan	Cotan	Tan	′
	49°		48°		47°		46°		

′	44° Tan	44° Cotan	′	′	44° Tan	44° Cotan	′	′	44° Tan	44° Cotan	′
0	.96569	1.03553	60	20	.97700	1.02355	40	40	.98843	1.01170	20
1	.96625	1.03493	59	21	.97756	1.02295	39	41	.98901	1.01112	19
2	.96681	1.03433	58	22	.97813	1.02236	38	42	.98958	1.01053	18
3	.96738	1.03372	57	23	.97870	1.02176	37	43	.99016	1.00994	17
4	.96794	1.03312	56	24	.97927	1.02117	36	44	.99073	1.00935	16
5	.96850	1.03252	55	25	.97984	1.02057	35	45	.99131	1.00876	15
6	.96907	1.03192	54	26	.98041	1.01998	34	46	.99189	1.00818	14
7	.96963	1.03132	53	27	.98098	1.01939	33	47	.99247	1.00759	13
8	.97020	1.03072	52	28	.98155	1.01879	32	48	.99304	1.00701	12
9	.97076	1.03012	51	29	.98213	1.01820	31	49	.99362	1.00642	11
10	.97133	1.02952	50	30	.98270	1.01761	30	50	.99420	1.00583	10
11	.97189	1.02892	49	31	.98327	1.01702	29	51	.99478	1.00525	9
12	.97246	1.02832	48	32	.98384	1.01642	28	52	.99536	1.00467	8
13	.97302	1.02772	47	33	.98441	1.01583	27	53	.99594	1.00408	7
14	.97359	1.02713	46	34	.98499	1.01524	26	54	.99652	1.00350	6
15	.97416	1.02653	45	35	.98556	1.01465	25	55	.99710	1.00291	5
16	.97472	1.02593	44	36	.98613	1.01406	24	56	.99768	1.00233	4
17	.97529	1.02533	43	37	.98671	1.01347	23	57	.99826	1.00175	3
18	.97586	1.02474	42	38	.98728	1.01288	22	58	.99884	1.00116	2
19	.97643	1.02414	41	39	.98786	1.01229	21	59	.99942	1.00058	1
20	.97700	1.02355	40	40	.98843	1.01170	20	60	1.00000	1.00000	0
′	45° Cotan	45° Tan	′	′	45° Cotan	45° Tan	′	′	45° Cotan	45° Tan	′

TABLE XV TRUE MASTER PARALLAX TABLE

$(H\text{-}h)$ ft.	$\Sigma\Delta p$ mm.	$(H\text{-}h)$ ft.	$\Sigma\Delta p$ mm.	$(H\text{-}h)$ ft.	$\Sigma\Delta p$ mm.	$(H\text{-}h)$ ft.	$\Sigma\Delta p$ mm.	$(H\text{-}h)$ ft.	$\Sigma\Delta p$ mm.
25,000	0	24,000	4.167	23,000	8.696	22,000	13.636	21,000	19.048
24,980	0.080	23,980	4.254	22,980	8.790	21,980	13.740	20,980	19.161
60	0.160	60	4.341	60	8.885	60	13.843	60	19.275
40	0.241	40	4.428	40	8.980	40	13.947	40	19.389
20	0.321	20	4.515	20	9.075	20	14.051	20	19.503
24,900	0.402	23,900	4.603	22,900	9.170	21,900	14.155	20,900	19.617
24,880	0.482	23,880	4.690	22,880	9.266	21,880	14.260	20,880	19.732
60	0.563	60	4.778	60	9.361	60	13.364	60	19.847
40	0.644	40	4.866	40	9.457	40	14.469	40	19.962
20	0.725	20	4.954	20	9.553	20	14.574	20	20.077
24,800	0.806	23,800	5.042	22,800	9.649	21,800	14.679	20,800	20.192
24,780	0.888	23,780	5.130	22,780	9.745	21,780	14.784	20,780	20.308
60	0.969	60	5.219	60	9.842	60	14.890	60	20.424
40	1.051	40	5.307	40	9.938	40	14.995	40	20.540
20	1.133	20	5.396	20	10.055	20	15.101	20	20.656
24,700	1.215	23,700	5.485	22,700	10.132	21,700	15.207	20,700	20.773
24,680	1.297	23,680	5.574	22,680	10.229	21,680	15.314	20,680	20.890
60	1.379	60	5.664	60	10.327	60	15.420	60	21.007
40	1.461	40	5.753	40	10.424	40	15.527	40	21.124
20	1.543	20	5.843	20	10.522	20	15.634	20	21.242
24,600	1.626	23,600	5.932	22,600	10.619	21,600	15.741	20,600	21.359
24,580	1.709	23,580	6.022	22,580	10.717	21,580	15.848	20,580	21.477
60	1.792	60	6.112	60	10.816	60	15.955	60	21.595
40	1.874	40	6.202	40	10.914	40	16.063	40	21.714
20	1.958	20	6.293	20	11.012	20	16.171	20	21.832
24,500	2.041	23,500	6.383	22,500	11.111	21,500	16.279	20,500	21.951
24,480	2.124	23,480	6.474	22,480	11.210	21,480	16.387	20,480	22.070
60	2.208	60	6.564	60	11.309	60	16.496	60	22.190
40	2.291	40	6.655	40	11.408	40	16.604	40	22.309
20	2.375	20	6.746	20	11.508	20	16.713	20	22.429
24,400	2.459	23,400	6.838	22,400	11.607	21,400	16.822	20,400	22.549
24,380	2.543	23,380	6.929	22,380	11.707	21,380	16.932	20,380	22.669
60	2.627	60	7.021	60	11.807	60	17.041	60	22.790
40	2.712	40	7.112	40	11.907	40	17.151	40	22.911
20	2.796	20	7.204	20	12.007	20	17.261	20	23.031
24,300	2.881	23,300	7.296	22,300	12.108	21,300	17.371	20,300	23.153
24,280	2.965	23,280	7.388	22,280	12.208	21,280	17.481	20,280	23.274
60	3.050	60	7.481	60	12.309	60	17.592	60	23.396
40	3.135	40	7.573	40	12.410	40	17.702	40	23.518
20	3.220	20	7.666	20	12.511	20	17.813	20	23.640
24,200	3.306	23,200	7.759	22,200	12.613	21,200	17.925	20,200	23.762
24,180	3.391	23,180	7.852	22,180	12.714	21,180	18.036	20,180	23.885
60	3.477	60	7.945	60	12.816	60	18.147	60	24.008
40	3.563	40	8.038	40	12.918	40	18.259	40	24.131
20	3.648	20	8.131	20	13.020	20	18.371	20	24.254
24,100	3.734	23,100	8.225	22,100	13.122	21,100	18.483	20,100	24.378
24,080	3.821	23,080	8.319	22,080	13.225	21,080	18.596	20,080	24.502
60	3.907	60	8.413	60	13.327	60	18.708	60	24.626
40	3.993	40	8.507	40	13.430	40	18.821	40	24.750
20	4.080	20	8.601	20	13.533	20	18.934	20	24.875

TABLE XV TRUE MASTER PARALLAX TABLE *(Continued)*

$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.
20,000	25.000	19,000	31.579	18,000	38.889	17,000	47.059	16,000	56.250
19,980	25.125	18,980	31.718	17,980	39,043	16,980	47.232	15,980	56.446
60	25.251	60	31.857	60	39.198	60	47.406	60	56.642
40	25.376	40	31.996	40	39.353	40	47.580	40	56.838
20	25.502	20	32.135	20	39.509	20	47.754	20	57.035
19,900	25.628	18,900	32.275	17,900	39.665	16,900	47.929	15,900	57.233
19,880	25,755	18.880	32,415	17,880	39.821	16,880	48.104	15,880	57.431
60	25.881	60	32.556	60	39.978	60	48.280	60	57.629
40	26.008	40	32.696	40	40.135	40	48.456	40	57.828
20	26.135	20	32.837	20	40.292	20	48.633	20	58.028
19,800	26.263	18,800	32.979	17,800	40.449	16,800	48.810	15,800	58.228
19,780	26.390	18,780	33.120	17,780	40.607	16,780	48.978	15,780	58.428
60	26.518	60	33.262	60	40.766	60	49.165	60	58.629
40	26.646	40	33.404	40	40.924	40	49.343	40	58.831
20	26.775	20	33.547	20	41.084	20	49.522	20	59.033
19,700	26.904	18,700	33.690	17,700	41.243	16,700	49.701	15,700	59.236
19,680	27,033	18,680	33.833	17,680	41.403	16,680	49.880	15,680	59.439
60	27.162	60	33.976	60	41.563	60	50.060	60	59.642
40	27.291	40	34.120	40	41.723	40	50.240	40	59.847
20	27.421	20	34.264	20	41.884	20	50.421	20	60.051
19,600	27.551	18,600	34.409	17,600	42.045	16,600	50.602	15,600	60.256
19,580	27.681	18,580	34.553	17,580	42.207	16,580	50.784	15,580	60.462
60	27.812	60	34.698	60	42.369	60	50.956	60	60.668
40	27.943	40	34.844	40	42.531	40	51.149	40	60.875
20	28.074	20	34.989	20	42.694	20	51.332	20	61.082
19,500	28.205	18,500	35.135	17,500	42.857	16,500	51.515	15,500	61.290
19,480	28.337	18,480	35.281	17,480	43.021	16,480	51.699	15,480	61.499
60	28.469	60	35.428	60	43.184	60	51.883	60	61.708
40	28.601	40	35.575	40	43.349	40	52.068	40	61.917
20	28.733	20	35.722	20	43.513	20	52.253	20	62.127
19,400	28.866	18,400	35.870	17,400	43.678	16,400	52.439	15,400	62.338
19,380	28,999	18,380	36.017	17,380	43.843	16,380	52.625	15,380	62.549
60	29.132	60	36.166	60	44.009	60	52.812	60	62.760
40	29.266	40	36.314	40	44.175	40	52.999	40	62.973
20	29.400	20	36.463	20	44.342	20	53.186	20	63.185
19,300	29.534	18,300	36.612	17,300	44.509	16,300	53.374	15,300	63.399
19,280	29.668	18,280	36.761	17,280	44.676	16,280	53.563	15,280	63.613
60	29.803	60	36.911	60	44.844	60	53.752	60	63.827
40	29.938	40	37.061	40	45.012	40	53.941	40	64.042
20	30.073	20	37.212	20	45.180	20	54.131	20	64.258
19,200	30.208	18,200	37.363	17,200	45.349	16,200	54.321	15,200	64.474
19,180	30.344	18,180	37.514	17,180	45.518	16,180	54.512	15,180	64.690
60	30.480	60	37.665	60	45.688	60	54.703	60	64.908
40	30.617	40	37.817	40	45.858	40	54.895	40	65.125
20	30.753	20	37.969	20	46.028	20	55.087	20	65.344
19,100	30.890	18,100	38.122	17,100	46.199	16,100	55.280	15,100	65.563
19,080	31.027	18,080	38.274	17,080	46.370	16,080	55.473	15,080	65.782
60	31.165	60	38.427	60	46.542	60	55.666	60	66.003
40	31.303	40	38.581	40	46.714	40	55.860	40	66.223
20	31.441	20	38.735	20	46.886	20	56.055	20	66.445

TABLE XV TRUE MASTER PARALLAX TABLE *(Continued)*

$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.
15,000	66.667	14,000	78.571	13,000	92.308	12,000	108.333	11,000	127.273
14,980	66.889	13,980	78.827	12,980	92.604	11,980	108.681	10,980	127.687
60	67.112	60	79.083	60	92.901	60	109.030	60	128.102
40	67.336	40	79.340	40	93.199	40	109.380	40	128.519
20	67.560	20	79.598	20	93.498	20	109.732	20	128.938
14,900	67.785	13,900	79.856	12,900	93.798	11,900	110.084	10,900	129.358
14,880	68.011	13,880	80.115	12,880	94.099	11,880	110.438	10,880	129.779
60	68.237	60	80.375	60	94.401	60	110.793	60	130.203
40	68.464	40	80.636	40	94.704	40	111.149	40	130.627
20	68.691	20	80.897	20	95.008	20	111.506	20	131.054
14,800	68.919	13,800	81.159	12,800	95.313	11,800	111.864	10,800	131.481
14,780	69.147	13,780	81.422	12,780	95.618	11,780	112.224	10,780	131.911
60	69.377	60	81.686	60	95.925	50	112.585	60	132.342
40	69.607	40	81.951	40	96.232	40	112.947	40	132.775
20	69.837	20	82.216	20	96.541	20	113.311	20	133.209
14,700	70.068	13,700	82.482	12,700	96.850	11,700	113.675	10,700	133.645
14,680	70.300	13,680	82.749	12,680	97.161	11,680	114.041	10,680	134.082
60	70.532	60	83.016	60	97.472	60	114.408	60	134.522
40	70.765	40	83.284	40	97.785	40	114.777	40	134.962
20	70.999	20	83.554	20	98.098	20	115.146	20	135.405
14,600	71.233	13,600	83.824	12,600	98.413	11,600	115.517	10,600	135.849
14,580	71.468	13,580	84.094	12,580	98.728	11,580	115.889	10,580	136.295
60	71.703	60	84.366	60	99.045	60	116.263	60	136.742
40	71.939	40	84.638	40	99.362	40	116.638	40	137.192
20	72.176	20	84.911	20	99.681	20	117.014	20	137.643
14,500	72.414	13,500	85.185	12,500	100.000	11,500	117.391	10,500	138.095
14,480	72.652	13,480	85.460	12,480	100.321	11,480	118.770	10,480	138.550
60	72.891	60	85.736	60	100.642	60	118.150	60	139.006
40	73.130	40	86.012	40	100.965	40	118.531	40	139.464
20	73.370	20	86.289	20	101.288	20	118.914	20	139.923
14,400	73.611	13,400	86.567	12,400	101.613	11,400	119.298	10,400	140.385
14,380	73.853	13,380	86.846	12,380	101.939	11,380	119.684	10,380	140.848
60	74.095	60	87.126	60	102.265	60	120.070	60	141.313
40	74.338	40	87.406	40	102.593	40	120.459	40	141.779
20	74.581	20	87.688	20	102.922	20	120.848	20	142.248
14,300	74.825	13,300	87.970	12,300	103.252	11,300	121.239	10,300	142.718
14,280	75.070	13,280	88.253	12,280	103.583	11,280	121.631	10,280	143.191
60	75.316	60	88.537	60	103.915	60	122.025	60	143.665
40	75.562	40	88.822	40	104.248	40	122.420	40	144.141
20	75.809	20	89.107	20	104.583	20	122.816	20	144.618
14,200	76.056	13,200	89.394	12,200	104.918	11,200	123.214	10,200	145.098
14,180	76.305	13,180	89.681	12,180	105.255	11,180	123.614	10,180	145.580
60	76.554	60	89.970	60	105.592	60	124.014	60	146.063
40	76.803	40	90.259	40	105.931	40	124.417	40	146.548
20	77.054	20	90.549	20	106.271	20	124.820	20	147.036
14,100	77.305	13,100	90.840	12,100	106.612	11,100	125.225	10,100	147.525
14,080	77.557	13,080	91.131	12,080	106.954	11,080	125.632	10,080	148.016
60	77.809	60	91.424	60	107.297	60	126.040	60	148.509
40	78.063	40	91.718	40	107.641	40	126.449	40	149.004
20	78.317	20	92.012	20	107.987	20	126.860	20	149.501

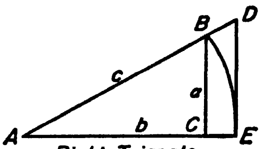
TABLE XV TRUE MASTER PARALLAX TABLE *(Continued)*

$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.
10,000	150.000	9,500	163.158	9,000	177.778	8,500	194.118	8,000	212.500
9,990	150.250	9,490	163.435	8,990	178.087	8,490	194.464	7,990	212.891
80	150.501	80	163.713	80	178.396	80	194.811	80	213.283
70	150.752	70	163.992	70	178.707	70	195.159	70	213.676
60	151.004	60	164.271	60	179.018	60	195.508	60	214.070
50	151.256	50	164.550	50	179.330	50	195.858	50	214.465
40	151.509	40	164.831	40	179.642	40	196.209	40	214.861
30	151.762	30	165.111	30	179.955	30	196.560	30	215.259
20	152.016	20	165.393	20	180.269	20	196.912	20	215.657
10	152.270	10	165.675	10	180.584	10	197.265	10	216.056
9,900	152.525	9,400	165.957	8,900	180.899	8,400	197.619	7,900	216.456
9,890	152.781	9,390	166.241	8,890	181.215	8,390	197.974	7,890	216.857
80	153.036	80	166.525	80	181.532	80	198.329	80	217.259
70	153.293	70	166.809	70	181.849	70	198.686	70	217.652
60	153.550	60	167.094	60	182.167	60	199.043	60	218.066
50	153.807	50	167.380	50	182.486	50	199.401	50	218.471
40	154.065	40	167.666	40	182.805	40	199.760	40	218.878
30	154.323	30	167.953	30	183.126	30	200.120	30	219.285
20	154.582	20	168.240	20	183.447	20	200.481	20	219.693
10	154.842	10	168.528	10	183.768	10	200.842	10	220.102
9,800	155.102	9,300	168.817	8,800	184.091	8,300	201.205	7,800	220.513
9,790	155.363	9,290	169.107	8,790	184.414	8,290	201.568	7,790	220.924
80	155.624	80	169.397	80	184.738	80	201.932	80	221.337
70	155.885	70	169.687	70	185.063	70	202.297	70	221.750
60	156.148	60	169.978	60	185.388	60	202.663	60	222.165
50	156.410	50	170.270	50	185.714	50	203.030	50	222.581
40	156.674	40	170.563	40	186.041	40	203.398	40	222.997
30	156.937	30	170.856	30	186.369	30	203.767	30	223.415
20	157.202	20	171.150	20	186.697	20	204.136	20	223.834
10	157.467	10	171.444	10	187.026	10	204.507	10	224.254
9,700	157.732	9,200	171.739	8,700	187,356	8,200	204.878	7,700	224 675
9,690	157.998	9,190	172.035	8,690	187.687	8,190	205.250	7,690	225.098
80	158.264	80	172.331	80	188.018	80	205.623	80	225.521
70	158.532	70	172.628	70	188.351	70	205.998	70	225.945
60	158.799	60	172.926	60	188.684	60	206.373	60	226.371
50	159.067	50	173.224	50	189.017	50	206.748	50	226.797
40	159.336	40	173.523	40	189.352	40	207.125	40	227.225
30	159.605	30	173.823	30	189.687	30	207.503	30	227.654
20	159.875	20	174.123	20	190.023	20	207.882	20	228.084
10	160.146	10	174.424	10	190.360	10	208.261	10	228.515
9,600	160.417	9,100	174.725	8,600	190.698	8,100	208.642	7,600	228.947
9,590	160.688	9,090	175.028	8,590	191.036	8,090	209.023	7,590	229.381
80	160.960	80	175.330	80	191.375	80	209.406	80	229.815
70	161.233	70	175.634	70	191.715	70	209.789	70	230.251
60	161.506	60	175.938	60	192.056	60	210.174	60	230.688
50	161.780	50	176.243	50	192.398	50	210.559	50	231.126
40	162.055	40	176.549	40	192.740	40	210.945	40	231.565
30	162.329	30	176.855	30	193.083	30	211.333	30	232.005
20	162.605	20	177.162	20	193.427	20	211.721	20	232.447
10	162.881	10	177.469	10	193.772	10	212.110	10	232.889

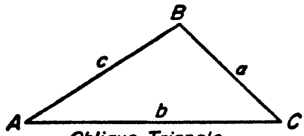
TABLE XV TRUE MASTER PARALLAX TABLE *(Continued)*

$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.	$(H-h)$ ft.	$\Sigma\Delta p$ mm.
7,500	233.333	7,000	257.143	6,500	284.615	6,000	316.667	5,500	354.545
7,490	233.778	6,990	257.654	6,490	285.208	5,990	317.362	5,490	355.373
80	234.225	80	258.166	80	285.802	80	318.060	80	356.204
70	234.672	70	258.680	70	286.399	70	318.760	70	357.038
60	235.121	60	259.195	60	286.997	60	319.463	60	357.875
50	235.570	50	259.712	50	287.597	50	320.168	50	358.716
40	236.022	40	260.231	40	288.199	40	320.875	40	359.559
30	236.474	30	260.750	30	288.802	30	321.585	30	360.405
20	236.927	20	261.272	20	289.408	20	322.297	20	361.255
10	237.382	10	261.795	10	290.016	10	323.012	10	362.107
7,400	237.838	6,900	262.319	6,400	290.625	5,900	323.729	5,400	352.963
7,390	238.295	6,890	262.845	6,390	291.236	5,890	324.448	5,390	363.822
80	238.753	80	263.372	80	291.850	80	325.170	80	364.684
70	239.213	70	263.901	70	292.465	70	325.894	70	365.549
60	239.674	60	264.431	60	293.082	60	326.612	60	366.418
50	240.136	50	264.964	50	293.701	50	327.350	50	367.290
40	240.599	40	265.497	40	294.322	40	328.082	40	368.165
30	241.064	30	266.032	30	294.945	30	328.816	30	369.043
20	241.530	20	266.569	20	295.570	20	329.553	20	369.925
10	241.997	10	267.107	10	296.197	10	330.293	10	370.810
7,300	242.466	6,800	267.647	6,300	296.825	5,800	331.034	5,300	371.698
7,290	242.936	6,790	268.189	6,290	297.456	5,790	331.779	5,290	372.590
80	243.407	80	268.732	80	298.089	80	332.526	80	373.485
70	243.879	70	269.276	70	298.724	70	333.276	70	374.383
60	244.353	60	269.822	60	299.361	60	334.028	60	375.285
50	244.828	50	270.370	50	300.000	50	334.783	50	376.190
40	245.304	40	270.920	40	300.641	40	335.540	40	377.099
30	245.781	30	271.471	30	301.284	30	336.300	30	378.011
20	246.260	20	272.024	20	301.929	20	337.063	20	378.927
10	246.741	10	272.578	10	302.576	10	337.828	10	379.846
7,200	247.222	6,700	273.134	6,200	303.226	5,700	338.596	5,200	380.769
7,190	247.705	6,690	273.692	6,190	303.877	5,690	339.367	5,190	381.696
80	248.189	80	274.251	80	304.531	80	340.141	80	382.625
70	248.675	70	274.813	70	305.186	70	340.917	70	383.559
60	249.162	60	275.375	60	305.844	60	341.696	60	384.496
50	249.650	50	275.940	50	306.504	50	342.478	50	385.437
40	250.140	40	276.506	40	307.166	40	343.262	40	386.381
30	250.631	30	277.074	30	307.830	30	344.050	30	387.329
20	251.124	20	277.644	20	308.497	20	344.840	20	388.281
10	251.617	10	278.215	10	309.165	10	345.633	10	389.237
7,100	252.113	6,600	278.788	6,100	309.836	5,600	346.429	5,100	390.196
7,090	252.609	6,590	279.363	6,090	310.509	5,590	347.227	5,090	391.159
80	253.107	80	279.939	80	311.184	80	348.029	80	392.126
70	253.607	70	280.518	70	311.862	70	348.833	70	393.097
60	254.108	60	281.098	60	312.541	60	349.640	60	394.071
50	254.610	50	281.679	50	313.223	50	350.450	50	395.050
40	255.114	40	282.263	40	313.907	40	351.264	40	396.032
30	255.619	30	282.848	30	314.594	30	352.080	30	397.018
20	256.125	20	283.436	20	315.282	20	352.899	20	398.008
10	256.633	10	284.025	10	315.973	10	353.721	10	399.002
								5,000	400.000

TABLE XVI. TRIGONOMETRIC FORMULAS.



Right Triangle



Oblique Triangle

RIGHT TRIANGLES

$$\sin A = \frac{a}{c} = \cos B \qquad \sec A = \frac{c}{b} = \operatorname{cosec} B$$

$$\cos A = \frac{b}{c} = \sin B \qquad \operatorname{cosec} A = \frac{c}{a} = \sec B$$

$$\tan A = \frac{a}{b} = \cot B \qquad \cot A = \frac{b}{a} = \tan B$$

$$a = c \sin A = c \cos B = b \tan A = b \cot B = \sqrt{c^2 - b^2}$$

$$b = c \cos A = c \sin B = a \cot A = a \tan B = \sqrt{c^2 - a^2}$$

$$c = \frac{a}{\sin A} = \frac{a}{\cos B} = \frac{b}{\sin B} = \frac{b}{\cos A}$$

OBLIQUE TRIANGLES

Given	Sought	Formulas
A, B, a	b, c	$b = \dfrac{a}{\sin A} \cdot \sin B \qquad c = \dfrac{a}{\sin A} \cdot \sin (A + B)$
A, a, b	B, c	$\sin B = \dfrac{\sin A}{a} \cdot b \qquad c = \dfrac{a}{\sin A} \cdot \sin C$
C, a, b	$\tfrac{1}{2}(A + B)$	$\tfrac{1}{2}(A + B) \doteq 90° - \tfrac{1}{2} C$
	$\tfrac{1}{2}(A - B)$	$\operatorname{Tan} \tfrac{1}{2}(A - B) = \dfrac{a - b}{a + b} \cdot \tan \tfrac{1}{2}(A + B)$
a, b, c	A	Given $s = \tfrac{1}{2}(a + b + c)$, then $\sin \tfrac{1}{2} A = \sqrt{\dfrac{(s - b)(s - c)}{bc}}$
		$\cos \tfrac{1}{2} A = \sqrt{\dfrac{s(s - a)}{bc}}$, $\tan \tfrac{1}{2} A = \sqrt{\dfrac{(s - b)(s - c)}{s(s - a)}}$
		$\sin A = 2 \dfrac{\sqrt{s(s - a)(s - b)(s - c)}}{bc}$
	Area	Area $= \sqrt{s(s - a)(s - b)(s - c)}$
C, a, b	Area	Area $= \tfrac{1}{2} ab \sin C$

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